Coupling Meshfree Methods and FEM for the Solution of the Incompressible Navier-Stokes Equations

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Meshfree Moving Least Squares and meshbased Finite Element shape functions are coupled in order to combine the advantages of each method. Meshfree methods are used in small parts of the domain, where a conforming mesh is difficult to maintain, e.g. due to large deformations. The FEM is used in the rest of the domain and enables an efficient solution of the underlying partial differential equations. The coupled formulation is successfully applied to flow simulations governed by the incompressible Navier-Stokes equations.

1 Introduction

Meshfree methods enable the approximation of partial differential equations based on a set of nodes without the need for an additional mesh, see [1, 5] for an overview of this comparably new class of methods. They have developed to be standard tools for problems where a conforming mesh is difficult to maintain. For example, in fluid mechanics this may be the case for flows involving moving and rotating objects. The advantage of meshfree methods comes at the price of being considerably more time-consuming than standard meshbased methods such as the Finite Element Method (FEM).

Therefore, we couple meshfree methods, in particular the Moving Least Squares (MLS) method [10], and FEM in order to combine the advantages of each method. Meshfree methods are used only in small parts of the domain where they are needed, whereas the FEM simulates the fluid in the rest of the domain.

In the following a brief outline of the MLS method and the coupling approaches of Belytschko *et al.* [4] and Huerta *et al.* [9] is given. The coupled formulations are applied successfully to the solution of the incompressible Navier-Stokes equations. However, it is found that convergence problems of the iterative solver may occur for small Reynolds numbers. A modification of the coupling approaches is needed in order to obtain shape functions that are more suited for stabilization.

2 Coupled meshfree/meshbased methods

Meshfree shape functions are often provided by the MLS methodology, see [5, 10] for a detailed discussion. The domain Ω is discretized by particles at $\mathbf{x}_i$ and corresponding pre-defined supports $\tilde{\Omega}_i$. Bell-shaped weighting functions $w_i\left(\mathbf{x}-\mathbf{x}_i\right)$ are defined inside $\tilde{\Omega}_i$. At each particle an approximation $\tilde{u}\left(\mathbf{x}\right) = \mathbf{p}^{\mathrm{T}}\left(\mathbf{x}\right)\mathbf{a}$ is applied, where $\mathbf{p}^{\mathrm{T}}\left(\mathbf{x}\right)$ is a vector of basis functions depending on the dimension and the desired order of consistency. The unknown coefficients $\mathbf{a}$ are obtained by a minimization of the least-squares error functional $J\left(\mathbf{a}\right) = \sum_i w_i\left(\mathbf{x}-\mathbf{x}_i\right)\left[u_i\left(\mathbf{x}\right) - \mathbf{p}^{\mathrm{T}}\left(\mathbf{x}_i\right)\mathbf{a}\right]^2$. Thereby, an expression of the kind $\tilde{u}\left(\mathbf{x}\right) = \mathbf{N}^{\mathrm{T}}\left(\mathbf{x}\right)\mathbf{u}$ is received, where $\mathbf{N}^{\mathrm{MLS}} = \mathbf{N}$ may be interpreted as the meshfree MLS shape functions and $\mathbf{u}$ are the unknown nodal values of the approximation. They may be determined —analogously to the FEM idea— with help of weighted residual methods.

In order to couple meshfree shape functions $\mathbf{N}^{\mathrm{MLS}}$ and standard meshbased shape functions $\mathbf{N}^{\mathrm{FEM}}$, the domain Ω is decomposed according to Fig. 1a) into an element subdomain Ω^{el} and a meshfree subdomain Ω^{MLS}. The union of elements along the boundary $\Omega^{\mathrm{el}} \cap \Omega^{\mathrm{MLS}}$ is called $\Omega^{\star}$ and plays an important role for the coupling procedures.

Two coupling approaches have been realized. The coupling approach introduced by Belytschko *et al.* in [4] works with ramp functions, that are 0 inside Ω^{FEM}, 1 in Ω^{MLS} and vary monotonically between 0 and 1 in $\Omega^{\star}$. The coupling approach of Huerta *et al.* [9] keeps the FEM shape functions unchanged in $\Omega^{\star}$. The desired order of consistency of the coupled shape functions is obtained by taking into account the FEM shape functions in $\Omega^{\star}$ by slightly modifying the MLS procedure.

3 Numerical Results

The coupled shape functions are applied to the solution of the incompressible Navier-Stokes equations. These equations are stabilized with the Streamline-Upwind and Pressure-Stabilizing Petrov-Galerkin (SUPG/PSPG) method [2, 3]. Note, that special attention is required for the stabilization of meshfree methods, see [6, 7].

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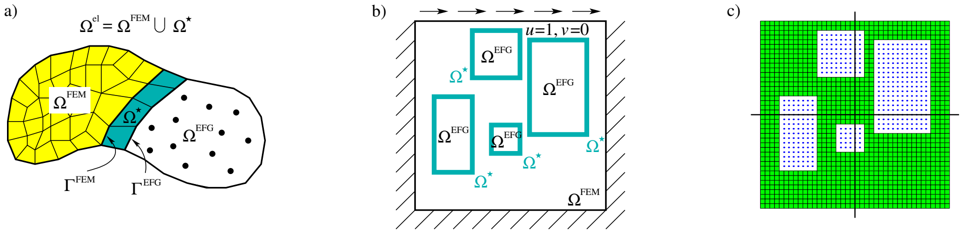

Fig. 1 a) Decomposition of the domain into Ω^{FEM}, Ω^{EFG} and $\Omega^\star$, b) problem statement and c) discretization of the driven cavity test case.

The driven cavity test case is a standard benchmark for fluid problems. Reference solutions are given in [8]. A flow inside a quadratic domain $\Omega = (0, 1) \times (0, 1)$ with no-slip boundary conditions on the left, right and lower wall develops under a shear flow applied on the upper boundary until a stationary solution is reached. Fig. 1b) gives an outline of the problem and c) shows a discretization with 41×41 nodes/particles.

The results of both approaches are almost identical for this test case and therefore, only the result for Huerta coupling is shown. In Figure 2c), planes for the velocities and pressure are shown over the domain. It may be seen, that the particles fit nicely into the FEM solution. Fig. 2a) and b) show the convergence against the reference solution along the center velocity profiles at a Reynolds number of $Re = 1000$. A comparison of the coupled solution with the pure FEM solution shows, that coupling does not adversely affect the accuracy of the solution.

It must be mentioned, that larger node/particle numbers result in convergence problems of the iterative solver of the non-linear Navier-Stokes equations at this Reynolds number. Therefore, a modification of the coupling approaches is required leading to coupled shape functions that are more suited for stabilization. This will be subject of further research.

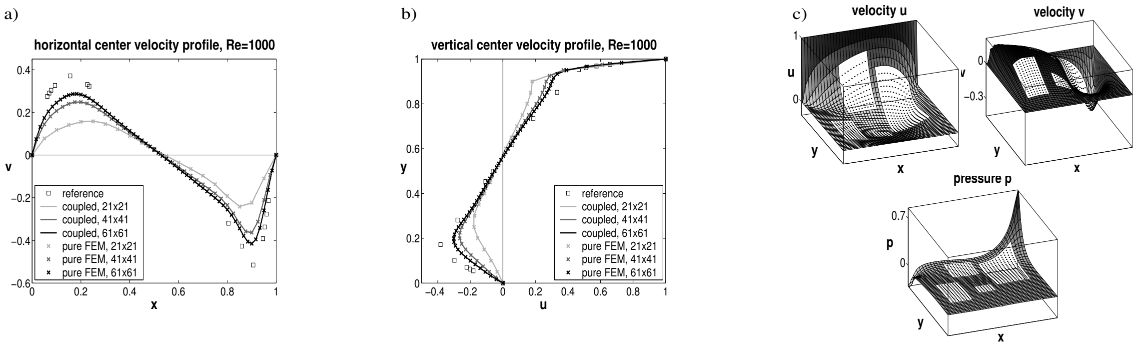

Fig. 2 a) and b) show the convergence against the reference solution, c) shows u-, v-, and p-planes for 41×41 nodes/particles.

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