Chapter 5

Beams and Frames

5.1 Introduction

Beams are slender members that are used for supporting transverse loading. Long horizontal members used in buildings and bridges, and shafts supported in bearings are some examples of beams. Complex structures with rigidly connected members are called frames and may be found in automobile and aeroplane structures and motion and force trasmitting machines and mechanisms.

A general horizontal beam with loading is shown in Figure 5.1(a). Figure 5.1(b) shows the deformation of the neutral axis.

Figure 5.2 below shows the bending stress distribution.

For small deflections,

$\sigma = -\frac{M}{I}y$

$\epsilon = -\frac{\sigma}{E}$

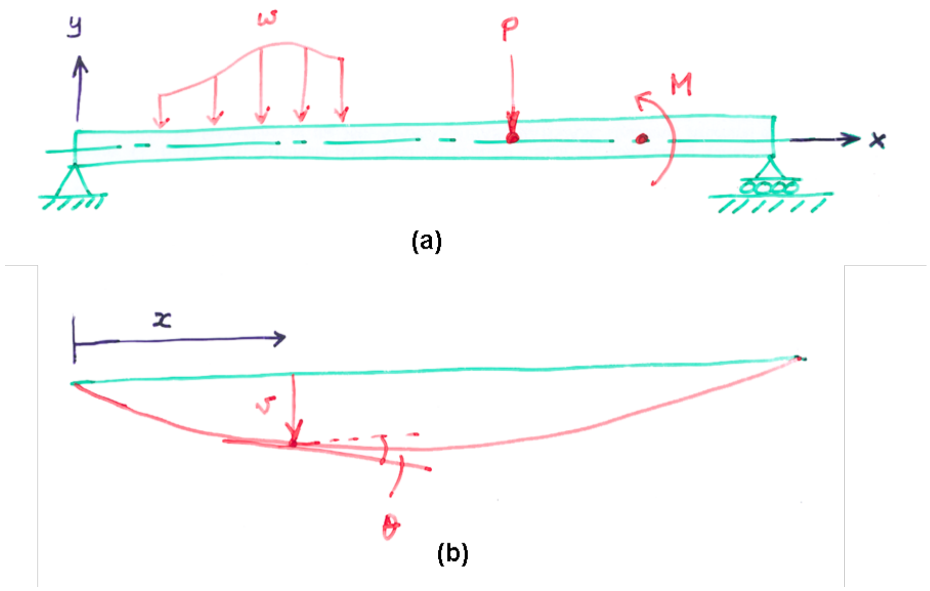


Figure 1: a) Beam loading (b) Deformation of the neutral axis

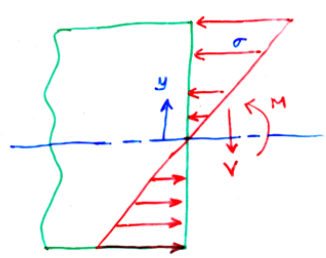


Figure 2:

$\frac{d^2v}{dx^2} = \frac{M}{EI}$

where $\sigma$is the normal stress, M is the bending moment at the section, v is the deflection of the centroidal axis at x and I is the moment of inertia of the section about the neutral axis (z-axis passing through the centroid).

5.2 Potential-Energy Approach

The potential energy of the beam is given by:

where p is the distributed load per unit length , P_m is the point load at point m, M_k is the moment of the couple applied at point k, v_m is the deflection at point m and v'_k is the slope at point k.

5.3 Finite Element Formulation

The beam is divided into elements, as shown in Figure 5.3. Each node has two degrees of freedom. Typically, the degrees of freedom of node i are Q_{2i-1} and Q_{2i}. The degree of freedom $Q_{2i-1}$is transverse displacement and Q_{2i} is slope or rotation.

The global displacement vector is represented by:

$Q = [Q_1, Q_2, \ldots, Q_{10}]^{\mathrm{T}}$

For a single element, the local degrees of freedom are represented by:

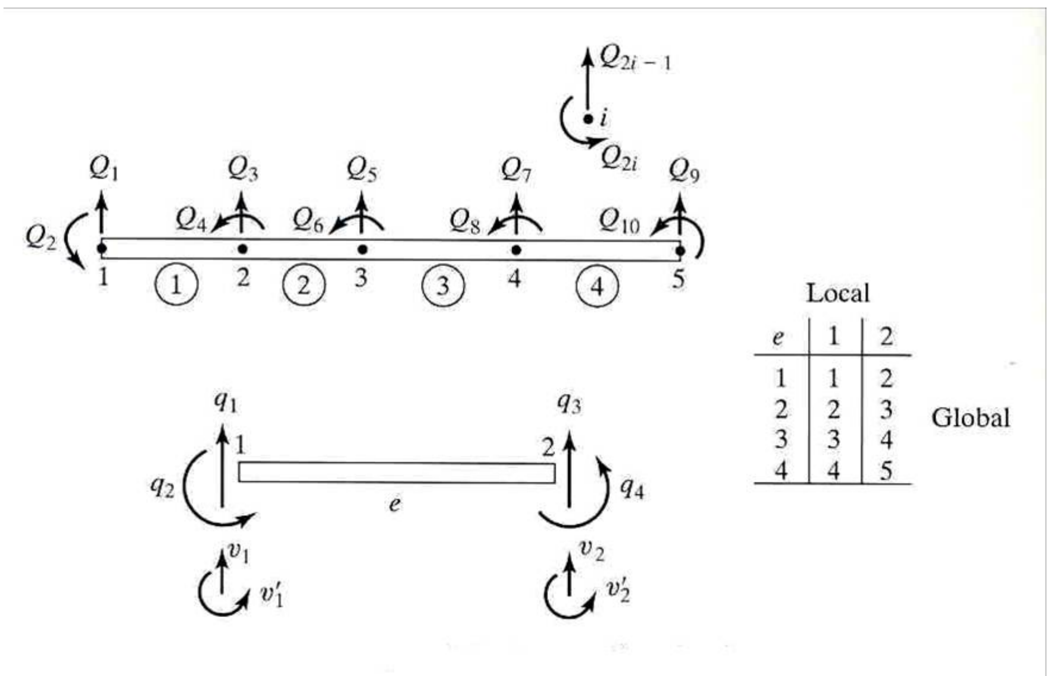
$\mathrm{q} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}}$

Hermite Shape Functions, which satisfies nodal value and slope continuity requirements is given by:

$Hi = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3$

The conditions given in the following table must be satisfied:

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1



Q_{2i-1}
i
Q_{2i}
Q_1 Q_3 Q_5 Q_7 Q_9
Q_2 Q_4 Q_6 Q_8 Q_{10}
1 2 3 4 5
1 2 3 4
Local
e 1 2
1 1 2
2 2 3
3 3 4
4 4 5
Global
q_1 q_2 q_3 q_4
1 2
e
v_1 v_1' v_2 v_2'

Figure 3:

The coefficients a_i, b_i, c_i and d_i can be easily obtained by imposing these conditions. Thus,

$H_1 = \frac{1}{4}\left(1-\xi\right)^2\left(2+\xi\right)$or $\frac{1}{4}\left(2-3\xi+\xi^3\right)$

$H_2 = \frac{1}{4}\left(1-\xi\right)^2\left(\xi+1\right)$or $\frac{1}{4}\left(1-\xi-\xi^2+\xi^3\right)$

$H_3 = \frac{1}{4}\left(1+\xi\right)^2\left(2-\xi\right)$or $\frac{1}{4}\left(2+3\xi+\xi^3\right)$

$H_3 = \frac{1}{4}\left(1+\xi\right)^2\left(\xi-1\right)$or $\frac{1}{4}\left(-1-\xi+\xi^2+\xi^3\right)$

Hermite shape functions can be used to write deflection, v in the form

$v\left(\xi\right) = H_1 v_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3 v_2 + H_4\left(\frac{dv}{d\xi}\right)_2$

The coordinate transform by the relationship:

$x = \frac{1-\xi}{2}x_1 + \frac{1-\xi}{2}x_2 = \frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\xi$

Since $\ell_e = x_2 - x_1$ is the length of the element, we have

$\frac{dx}{d\xi} = \frac{\ell_e}{2}$

The chain rule, $\frac{dx}{d\xi} = \left(\frac{dv}{dx}\right)\left(\frac{dx}{d\xi}\right)$gives us

$\frac{dv}{d\xi} = \left(\frac{dv}{dx}\right)\left(\frac{dx}{d\xi}\right)$

Noting that $\frac{dv}{dx}$evaluated at nodes 1 and 2 is q_2 and q_4, respectively, we have

$v\left(\xi\right) = H_1 q_1 + \frac{\ell_e}{2}H_2 q_2 + H_3 q_3 + \frac{\ell_e}{2}H_4 q_4$

which may be denoted as

$v = \mathrm{Hq}$

where

$H = \left(H_1,\ \frac{\ell_e}{2}H_2,\ H_3,\ \frac{\ell_e}{2}H_4\right)$

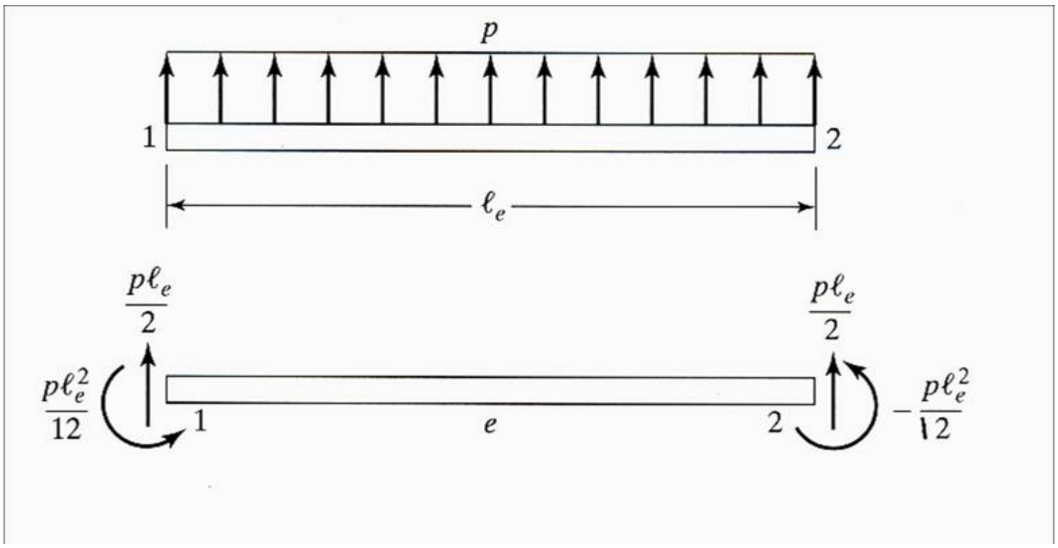


Figure 4:

Element strain energy given by:

$U_e = \frac{1}{2} EI \int_e \left(\frac{d^2v}{dx^2}\right)^2 dx$

Where the element stiffness matrix is:

$$k_e = \frac{EI}{\ell e} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}$$

5.4 Load Vector

Referring to Figure 5.4 above, the equivalent load on an element, is given by:

$\mathrm{f}_e = \left[\frac{p\ell_e}{2}, \frac{p\ell_e^2}{12}, \frac{p\ell_e}{2}, \frac{-p\ell_e^2}{12}\right]^{\mathrm{T}}$

5.5 Shear Force and Bending Moment

Using the bending moment and shear force equations,

$M = EI\frac{d^2v}{dx^2}, \qquad V = \frac{dM}{dx} \quad and \quad v = \text{Hq}$

we get the element bending moment and shear force:

$M = \frac{EI}{\ell_e^2}\left[6\xi q_1 + (3\xi - 1)\,\ell_e q_2 - 6\xi q_3 + (3\xi + 1)\,\ell_e q_4\right]$

$V = \frac{6EI}{\ell_e^3}\left[2q_1 + \ell_e q_2 - 2q_3 + \ell_e q_4\right]$

Reaction force equation is given by: R = KQ − F

Example:

For the beam and loading in Figure 5.5 below, determine (1) the slope at 2 and 3 and (2) the vertical deflection at the midpoint of the distributed load:

Solution:

We consider the two elements formed by the three nodes.Displacements $Q_1, Q_2, Q_3$and $Q_5$are constrained to be zero, and $Q_4$and Q_6need to be found. Since the lengths and sections are equal, the element stiffness matrices are calculated as follows:

$$k_e = \frac{EI}{\ell e}\begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}$$

Element stiffness matrices for element 1 and 2 are given by:

$$k^1 = k^2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$e = 1 \qquad\qquad Q_1 \quad Q_2 \quad Q_3 \quad Q_4$

$e = 2 \qquad\qquad Q_3 \quad Q_4 \quad Q_5 \quad Q_6$

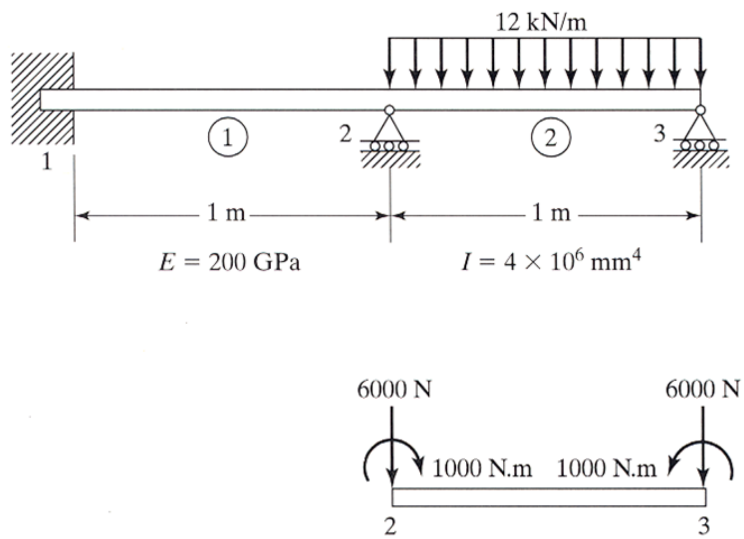

12 kN/m
1
2
3
1 m
1 m
E = 200 GPa
I = 4 × 10^6 mm^4
6000 N
6000 N
1000 N.m
1000 N.m
2
3

Figure 5:

Refer to the Figure 5.4, the global applied loads are $F_4 = -1000\,\mathrm{Nm}$ and $F_4 = +1000\,\mathrm{Nm}$ obtained from $\frac{p\ell_e^2}{12}$. We use here the elimination approach. Using the connectivity, we obtain the global stiffness matrix after elimination:

$$\mathrm{K} = \begin{bmatrix} k_{44}^1 + k_{22}^2 & k_{24}^2 \\ k_{42}^2 & k_{44}^2 \end{bmatrix} = 8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

Because $Q_1, Q_2, Q_3$and $Q_5$are zero, the set of equation is given by:

$$8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ +1000 \end{Bmatrix}$$

The solution is:

$$\begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -2.679 \times 10^{-4} \\ 4.464 \times 10^{-4} \end{Bmatrix}$$

For element 2, $q_1 = 0$, $q_2 = Q_4$, $q_3 = 0$ and $q_4 = Q_6$. To get vertical deflection at the midpoint, use $v = \mathrm{Hq}$ at $\xi = 0$.

$$v = 0 + \tfrac{\ell_e}{2} H_2 Q_4 + 0 + \tfrac{\ell_e}{2} H_4 Q_6$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\left(-2.679 \times 10^{-4}\right) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{4}\right)\left(4.464 \times 10^{-4}\right)$$

$$= -8.93 \times 10^{-5}\,\mathrm{m}$$

$$= -0.0893\,\mathrm{mm}$$

Problem above can be solved using BEAM program:

Input data for BEAM program:

```
<< BEAM ANALYSIS >>
EXAMPLE 8.1
        NN           NE        NM       NDIM          NEN        NDN
         3            2         1          1            2          2
        ND           NL      NMPC
         4            4         0
   NODE#          X-COORD
         1            0
         2         1000
         3         2000
       EL#           N1        N2       MAT#Mom_Inertia
         1            1         2          1     4.00E+06
         2            2         3          1     4.00E+06
      DOF#        Displ.
         1            0
         2            0
         3            0
         5            0
      DOF#         LOAD
         3        -6000
         4    -1.00E+06
         5        -6000
         6     1.00E+06
      MAT#            E
         1       200000
B1         I              B2       J          B3
```

Results from BEAM program:

```
Results from Program BEAM
EXAMPLE 8.1
 Node#  Displ.     Rotation
     1  4.02E-11    1.3392E-08
     2  -2.5E-10   -0.00026786
     3  -1.6E-10    0.00044643
 DOF#  Reaction
     1  -1285.67
     2   -428535
     3  8142.802
     5  5142.866
```

5.6 Plane Frames

Here, we consider plane structures with rigidly connected members. These members will be similar to the beams except that axial loads and axial deformations are present. The

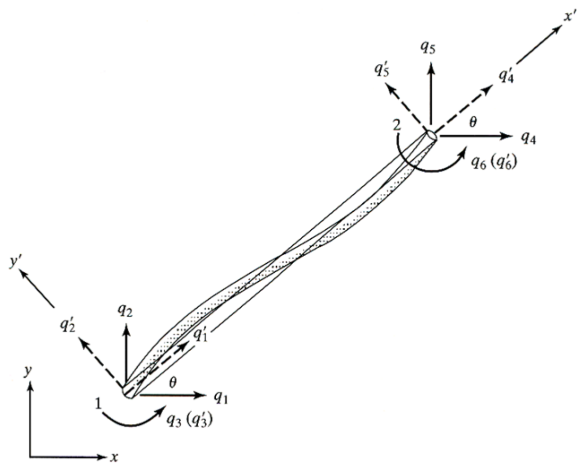


Figure 6:

elements also have different orientations. Figure 5.6 below shows a frame element. For each node, we have two displacements and a rotational deformation.

The nodal displacement vector in the global system is given by:

$q = [q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6]^{\mathrm{T}}$

The nodal displacement vector in the local system is given by:

$q' = [q'_1,\ q'_2,\ q'_3,\ q'_4,\ q'_5,\ q'_6]^{\mathrm{T}}$

Local-global transformation is:

$\mathrm{q}' = \mathrm{Lq}_1$

where :

$$\mathrm{L} = \begin{bmatrix} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

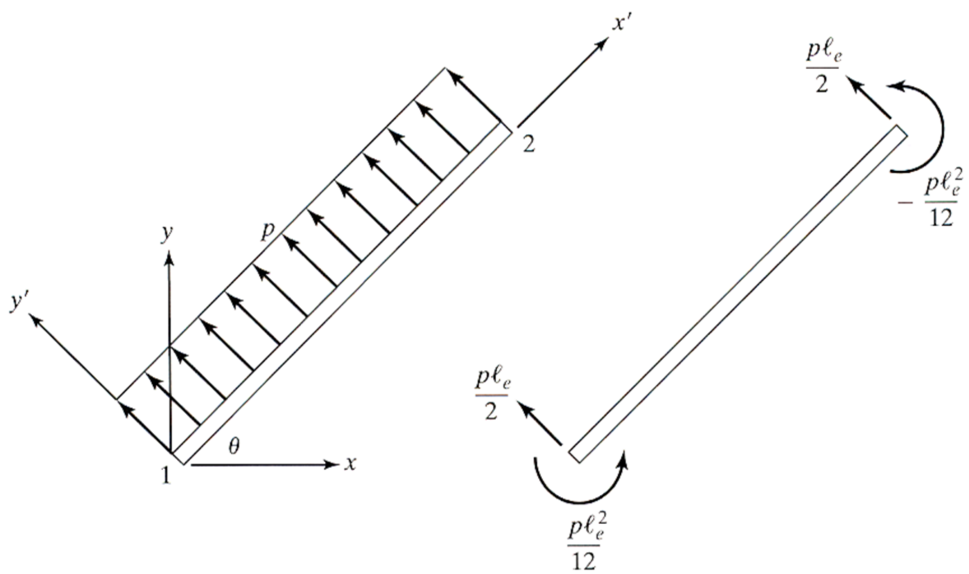


Figure 7:

$\ell = cos\,\theta,\ m = sin\theta$

$$k'_e = \begin{bmatrix} \frac{EA}{\ell_e} & 0 & 0 & -\frac{EA}{\ell_e} & 0 & 0 \\ 0 & \frac{12EI}{\ell_e^3} & \frac{6EI}{\ell_e^2} & 0 & -\frac{12EI}{\ell_e^3} & \frac{6EI}{\ell_e^2} \\ 0 & \frac{6EI}{\ell_e^2} & \frac{4EI}{\ell_e} & 0 & \frac{6EI}{\ell_e^2} & \frac{2EI}{\ell_e} \\ -\frac{EA}{\ell_e} & 0 & 0 & \frac{EA}{\ell_e} & 0 & 0 \\ 0 & -\frac{12EI}{\ell_e^3} & \frac{6EI}{\ell_e^2} & 0 & \frac{12EI}{\ell_e^3} & -\frac{6EI}{\ell_e^2} \\ 0 & \frac{6EI}{\ell_e^2} & \frac{2EI}{\ell_e} & 0 & -\frac{6EI}{\ell_e^2} & \frac{4EI}{\ell_e} \end{bmatrix}$$

From the element strain energy or in Galerkin's approach, we recognize the element stiffness matrix in global coordinates to be:

$\mathrm{k^e = L^T k'_e L}$

If there is distributed load on a member, as shown in Figure 5.7, we have:

$\mathrm{q'^T f' = q^T L^T f'}$

where $\mathrm{f}' = \left[0,\ \frac{p\ell_e}{2},\ \frac{p\ell_e^2}{12},\ 0,\ \frac{p\ell_e}{2},\ -\frac{p\ell_e^2}{12}\right]^{\mathrm{T}}$

The nodal loads due to the distributed load, p are given by:

$\mathrm{f} = \mathrm{L}^{\mathrm{T}}\mathrm{f}'$

The system of equations:

$\mathrm{KQ} = \mathrm{F}$

where penalty approach is applied to solve the equations.

Example:

Detremine the displacements and rotations of the joints for the portal frame shown in Figure 5.8 below.

Solution:

The connectivity is as follows:

Element No.	Node	
	1	2
1	1	2
2	3	1
3	4	2

Element Stiffness

Element 1: Noting that, $k^1 = k'^1$, we find that

$$k^1 = 10^4 \times \begin{array}{c} \begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array} \\ \left[\begin{array}{cccccc} 141.7 & 0 & 0 & -141.7 & 0 & 0 \\ 0 & 0.784 & 56.4 & 0 & -0.784 & 56.4 \\ 0 & 56.4 & 5417 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 141.7 & 0 & 0 \\ 0 & -0.784 & -56.4 & 0 & 0.784 & -56.4 \\ 0 & 56.4 & 2708 & 0 & -56.4 & 5417 \end{array}\right] \end{array} \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array}$$

Elements 2 and 3:

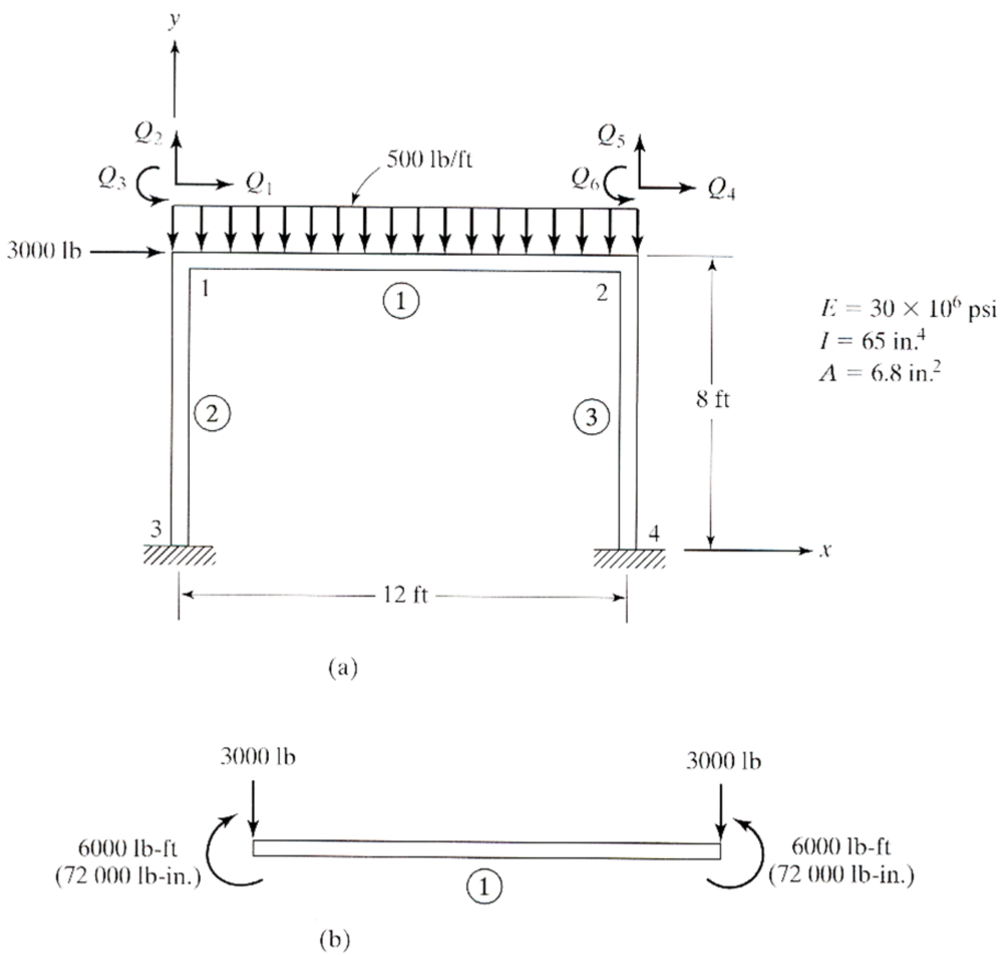


Figure 8: (a) Portal frame (b) Equivalent load for element 1

Local element stiffnesses for elements 2 and 3 are obtained by substituting for E, A, I and ℓ_2 in matrix k'^e. We get:

$$k'^3 = k'^2 = 10^4 \times \begin{bmatrix} 212.5 & 0 & 0 & -212.5 & 0 & 0 \\ 0 & 2.65 & 127 & 0 & -2.65 & 127 \\ 0 & 127 & 8125 & 0 & -127 & 4063 \\ -212.6 & 0 & 0 & 212.5 & 0 & 0 \\ 0 & -2.65 & -127 & 0 & 2.65 & -127 \\ 0 & 127 & 4063 & 0 & .127 & 8125 \end{bmatrix}$$

Transformation matrix L:

We have noted that for element 1, $k^1 = k'^1$. For elements 2 and 3, which are oriented similarly with respect to the x and y axes, we have $\ell = 0$ and $m = 1$. Then,

$$\mathrm{L}^1 = \mathrm{L}^2 = \mathrm{L} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Noting that $\mathrm{k}^2 = \mathrm{L}^{\mathrm{T}} \mathrm{k}'^2 \mathrm{L}$, we get,

$$k^2 = 10^4 \times \begin{array}{c} \begin{matrix} Q_7 & Q_8 & Q_9 & Q_1 & Q_2 & Q_3 \end{matrix} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ 127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array} \begin{matrix} \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_1 \\ Q_2 \\ Q_3 \end{matrix}$$

$$k^3 = 10^4 \times \begin{array}{c} \begin{matrix} Q_{10} & Q_{11} & Q_{12} & Q_4 & Q_5 & Q_6 \end{matrix} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array} \begin{matrix} \\ Q_{10} \\ Q_{11} \\ Q_{12} \\ Q_4 \\ Q_5 \\ Q_6 \end{matrix}$$

By adding and rearranging the element matrices, and $Q_7 \quad Q_8 \quad Q_9 \quad Q_{10} \quad Q_{11} \quad Q_{12}$ is zero, then the structural stiffness matrix K is given by:

$$
\mathrm{K} = 10^4 \times
\begin{array}{c}
\begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array} \\
\left[\begin{array}{cccccc}
144.3 & 0 & 127 & -141.7 & 0 & 0 \\
0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\
127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\
-141.7 & 0 & 0 & 144.3 & 0 & 127 \\
0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\
0 & 56.4 & 2708 & 127 & -56.4 & 13542
\end{array}\right]
\end{array}
\begin{array}{c} \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array}
$$

From Figure 5.7(b), load vector can be written as:

$$
\mathrm{F} = \left\{ \begin{array}{c} 3000 \\ -3000 \\ -72000 \\ 0 \\ -3000 \\ +72000 \end{array} \right\}
$$

The finite element equation is given by:

$\mathrm{KQ} = \mathrm{F}$

$$
10^4 \times \left[\begin{array}{cccccc}
144.3 & 0 & 127 & -141.7 & 0 & 0 \\
0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\
127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\
-141.7 & 0 & 0 & 144.3 & 0 & 127 \\
0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\
0 & 56.4 & 2708 & 127 & -56.4 & 13542
\end{array}\right]
\left\{ \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array} \right\}
= \left\{ \begin{array}{c} 3000 \\ -3000 \\ -72000 \\ 0 \\ -3000 \\ +72000 \end{array} \right\}
$$

By solving the equation, we have:

$$
\mathrm{Q} = \left\{ \begin{array}{c} 0.092\, in \\ -0.00104\, in \\ -0.00139\, rad \\ 0.0901\, in \\ -0.0018\, in \\ -3.88 \times 10 - 5\, rad \end{array} \right\}
$$

Problem above can be solved using computer by using Frame2D program:

<<2-D FRAME ANALYSIS >>						
EXAMPLE 8.2						
NN	NE	NM	NDIM	NEN	NDN	
4	3	1	2	2	3	
ND	NL	NMPC				
6	1	0				
Node#	X	Y				
1	0	96				
2	144	96				
3	0	0				
4	144	0				
Elem#	N1	N2	Mat#	Area	Inertia	Distr_Load
1	2	1	1	6.8	65	41.6666
2	3	1	1	6.8	65	0
3	4	2	1	6.8	65	0
DOF#	Displ.					
7	0					
8	0					
9	0					
10	0					
11	0					
12	0					
DOF#	Load					
1	3000					
MAT#	E					
1	3.00E+07					
B1	i	B2	j	B3	(Multi-point constr. B1*Qi+B2*Qj=B3)	

Figure 9:

Input data for Frame2D program:

Result from Frame2D program:

5.7 Three-dimensional Frames

Three dimensional frames, also called as space frames, are frequently encountered in the analysis of multistory buidings. They are also to be found in the modeling of car body and bicycle frames. A typical three-dimensional frame is shown in Figure 5.8 below. Each node has six degree of freedom (as opposed to only three dofs in a plane frame).

Local coordinate system is given by:

$$q\prime = \Big[\underbrace{q'_1, q'_2, q'_3}_{\text{Translasi nod 1}} \quad , \quad \underbrace{q'_4, q'_5, q'_6}_{\text{Putaran nod 1}} \quad , \quad \underbrace{q'_7, q'_8, q'_9}_{\text{Translasi nod 2}} \quad , \quad \underbrace{q'_{10}, q'_{11}, q'_{12}}_{\text{Putaran nod 2}}\Big]$$

Global coordinate system is given by:

```
Results from Program Frame2D
EXAMPLE 8.2
         Node#          X-Displ          Y-Displ   Z-Rotation
             1          0.09177         -0.00104     -0.00139
             2         0.090122         -0.00179     -3.9E-05
             3         4.92E-10         -1.6E-09     -4.4E-08
             4         1.72E-09         -2.8E-09     -8.3E-08
Member End-Forces
Member#                       1
        2334.2         -798.836         -39254.6
       -2334.2          798.836         -75777.8
Member#                       2
      2201.159         665.7996         60138.81
      -2201.16           -665.8         3777.949
Member#                       3
      3798.831           2334.2         112828.8
      -3798.83          -2334.2         111254.4
DOF#                Reaction
             7           -665.8
             8         2201.159
             9         60138.81
            10          -2334.2
            11         3798.831
            12         112828.8
```

Figure 10:

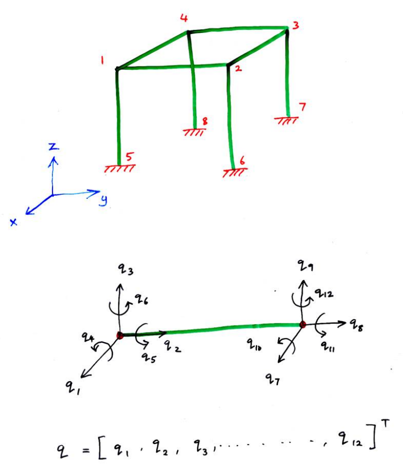


Figure 11:

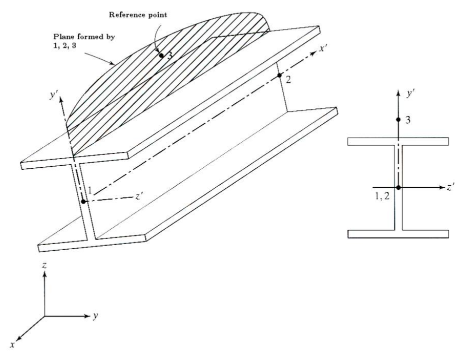


Figure 12:

$$q = \left[\underbrace{q_1', q_2', q_3', q_4', q_5', q_6', q_7', q_8', q_9', q_{10}', q_{11}', q_{12}'}_{Vektor\ anjakan\ dalam\ sistem\ sejagat\ x,y\ dan\ z}\right]$$

Local coordinate system is established with the use of three points. Points 1 and 2 are the ends of the element; the x'-axis is along the line from point 1 to point 2, just in the case of two-dimensional frames. Point 3 is any reference point not laying along the line joining points 1 and 2. The y'-axis is to lie in the plane defined by points 1, 2 and 3. The z'-axis is then automatically defined from the fact that x', $y'$and z' form a right handed system. We note that y', $z'$are the principal axes of the cross section with $I_{y\prime}$ and $I_{z\prime}$ are the principal moments of inertia. The cross-sectional properties are specified by four parameters: area A, and moments of inertia $I_{y\prime}$, $I_{z\prime}$, and J. The product GJ is the torsional stiffness, where $G =$ shear modulus. For circular or tubular cross section, J is the polar moment of inertia. For other cross-sectional shapes, such as an I-section, the torsional stiffness is given in the strength of materials texts.

The (12x12) element stiffness matrix, in the local coordinate system is given by:

$$k' = \begin{bmatrix} AS & 0 & 0 & 0 & 0 & 0 & -AS & 0 & 0 & 0 & 0 & 0 \\ & a_z^{'} & 0 & 0 & 0 & b_z^{'} & 0 & -a'_z & 0 & 0 & 0 & b'_z \\ & & a'_y & 0 & -b_y & 0 & 0 & 0 & -a'_y & 0 & b'_y & 0 \\ & & & TS & 0 & 0 & 0 & 0 & 0 & -TS & 0 & 0 \\ & & & & c'_y & 0 & 0 & 0 & b'_y & 0 & d'_y & 0 \\ & & & & & c'_z & 0 & -b'_z & 0 & 0 & 0 & d'_z \\ & & & & & & AS & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & a'_z & 0 & 0 & 0 & -b'_z \\ & & & & & & & & c'_y & 0 & b'_y & 0 \\ & & & & & & & & & TS & 0 & 0 \\ & & & & & & & & & & c'_y & 0 \\ symmetry & & & & & & & & & & & c'_z \end{bmatrix}$$

where:

$AS = \frac{EA}{l_e}$, $l_e = element\, length$, $TS = \frac{GJ}{l_e}$, $a'_z = \frac{12EI_{z'}}{l_e^3}$, $b'_z = \frac{6EI}{l_e^3}$

$c_z^{'} = \frac{4EI_{z^{'}}}{l_e}$, $d_z^{'} = \frac{2EI_{z^{'}}}{l'_e}$,$a_y^{'} = \frac{12EI_{y^{'}}}{l_e^3}$ and so on.

The global-local transformation matrix is given by:

q = Lq

The (12x12) transformation matrix L is defined as:

$$L = \begin{bmatrix} l_1 & m_1 & n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_2 & m_2 & n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_3 & m_3 & n_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 & m_1 & n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_2 & m_2 & n_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_3 & m_3 & n_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_1 & m_1 & n_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_2 & m_2 & n_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_3 & m_3 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_1 & m_1 & n_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_2 & m_2 & n_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_3 & m_3 & n_3 \end{bmatrix}$$

or

$$\mathrm{L} = \begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}$$

where

$$\lambda = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

l_1, $m_1$and $n_1$are the cosines of the angles between the x'-axis and the global x, y and z axes. Similarly l_2, $m_2$and n_2 are the cosines of the angles between the y'-axis and the global x, y and z axes. and are the cosines of the angles between the axes and the global x, y and z axes.

These direction cosines and hence the λ matrix are obtainable from the coordinates of the points 1, 2 and 3 as follows:

$l_1 = \frac{x_2 - x_1}{l_e}$ with $l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$m_1 = \frac{y_2 - y_1}{l_e}$

$n_1 = \frac{z_2 - z_1}{l_e}$

The element stiffness matrix in global coordinates is:

$\mathrm{k} = \mathrm{L}^{\mathrm{T}}\mathrm{k}'\mathrm{L}$

If a distributed load with components w_y' and w_z' (units of force/unit length) is applied on the element, then the equivalent loads at the ends of the member are:

$$\mathrm{f}' = \left[0, \frac{w_{y'} l_e}{2}, \frac{w_{z'} l_e}{2}, 0, \frac{-w_{z'} l_e^2}{12}, \frac{w_{y'} l_e^2}{12}, 0, \frac{w_{y'} l_e}{2}, \frac{w_{z'} l_e}{2}, 0, \frac{-w_{z'} l_e^2}{12}, \frac{w_{y'} l_e^2}{12}\right]^{\mathrm{T}}$$

These loads are transferred into global components by $\mathrm{f} = \mathrm{L}^{\mathrm{T}}\mathrm{f}'$. After enforcing boundary conditions and solving the system equations $\mathrm{KQ} = \mathrm{F}$, we can compute the member end forces from

$R' = k'q' + fixed\,end\,reactions$

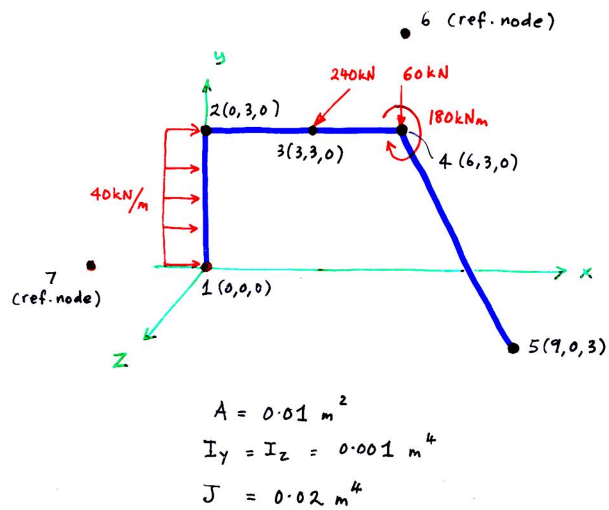


Figure 13:

where the fixed-end reactions are the negative of the f′ vector and are only associated with those elements having distributed loads acting on them.

Example

Figure 5.10 shows a three-dimensional frame subjected to various loads. By running Frame3D program, find the maximum bending moments in the structure.

Input data for Frame3D program:

```
<<3-D Frame Analysis >>
EXAMPLE
     NN        NE        NM      NDIM    NEN    NDN    NNREF
      5         4         1         3      2      6        2
     ND        NL      NMPC
     12         3         0
  Node#         X         Y         Z
      1         0         0         0
      2         0         3         0
      3         3         3         0
      4         6         3         0
      5         9         0         3
      6         6         6         0
      7        -3         0         0
  Elem#        N1        N2    Ref_Pt   Mat#   Area      Iy      Iz       J    UDLy'   UDLz'
      1         1         2         7      1   0.01   0.001   0.001   0.002   -40000       0
      2         2         3         6      1   0.01   0.001   0.001   0.002        0       0
      3         3         4         6      1   0.01   0.001   0.001   0.002        0       0
      4         4         5         6      1   0.01   0.001   0.001   0.002        0       0
   DOF#    Displ.
      1         0
      2         0
      3         0
      4         0
      5         0
      6         0
     25         0
     26         0
     27         0
     28         0
     29         0
     30         0
   DOF#      Load
     15    240000
     20    -60000
     24   -180000
   MAT#         E         G
      1  2.00E+11  8.00E+10
```

Result from Frame3D program:

```
Results from Program Frame3D
EXAMPLE
Node#       X-Displ     Y-Displ     Z-Displ    X-Rot      Y-Rot       Z-Rot
1           3.13E-09    1.97E-09    9.90E-09   2.76E-08   -7.14E-09   5.35E-09
2           -1.87E-03   3.94E-05    5.31E-03   2.55E-03   -1.79E-03   1.11E-03
3           -1.99E-03   3.14E-03    9.84E-03   2.03E-03   -2.45E-04   7.62E-04
4           -2.10E-03   3.43E-03    6.24E-03   1.50E-03   1.84E-03    -7.66E-04
5           5.87E-09    -6.47E-09   8.10E-09   6.98E-09   8.43E-09    -1.10E-09
Member End-Forces
Member#     1           End Forces
-2.63E+04   -1.83E+04   -1.32E+05   9.53E+04   3.68E+05   -1.01E+05
2.63E+04    1.83E+04    1.32E+05    9.53E+04   2.80E+04   4.64E+04
Member#     2           End Forces
7.83E+04    -2.63E+04   -1.32E+05   2.80E+04   9.53E+04   -1.64E+04
-7.83E+04   2.63E+04    1.32E+05    2.80E+04   3.01E+05   -6.25E+04
Member#     3           End Forces
7.83E+04    -2.63E+04   1.08E+05    2.80E+04   3.01E+05   6.25E+04
-7.83E+04   2.63E+04    -1.08E+05   2.80E+04   2.33E+04   -1.41E+05
Member#     4           End Forces
1.57E+05    5.60E+03    2.10E+04    1.96E+04   1.47E+04   -4.71E+04
-1.57E+05   -5.60E+03   -2.10E+04   1.96E+04   1.24E+05   7.62E+04
DOF#        Reaction
1           -4.17E+04
2           -2.63E+04
3           -1.32E+05
4           -3.68E+05
5           9.53E+04
6           -7.13E+04
25          -7.83E+04
26          8.63E+04
27          -1.08E+05
28          -9.31E+04
29          -1.12E+05
30          1.47E+04
```

5.8 Case Study/Engineering Application:

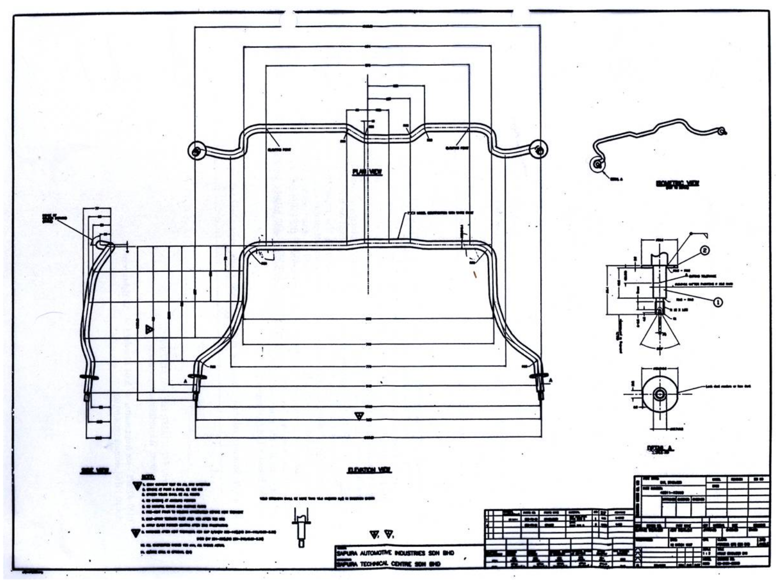
ANALYSIS OF THE STABILIZER BAR UNDER LOADING CONDITION

1-Objective:

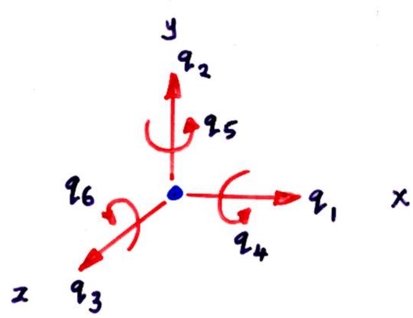
To run simulation on the stabilizer bar under various loading conditions.

2-Introduction:

The stabilizer bar structure is shown in Figure 5.12 below.



The stabilizer bar is modeled as a three dimensional problem with linear elastic material properties. Each node has 6 degrees of freedoms along x, y and z axes.



$q = [q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6]^{\mathrm{T}}$

The local stiffness matrix, k (12 x 12 matrix) is a function of element stiffness (EA/l), torsional stiffness $(GJ/l)$and moment of inertia I. Global stiffness matrix is:

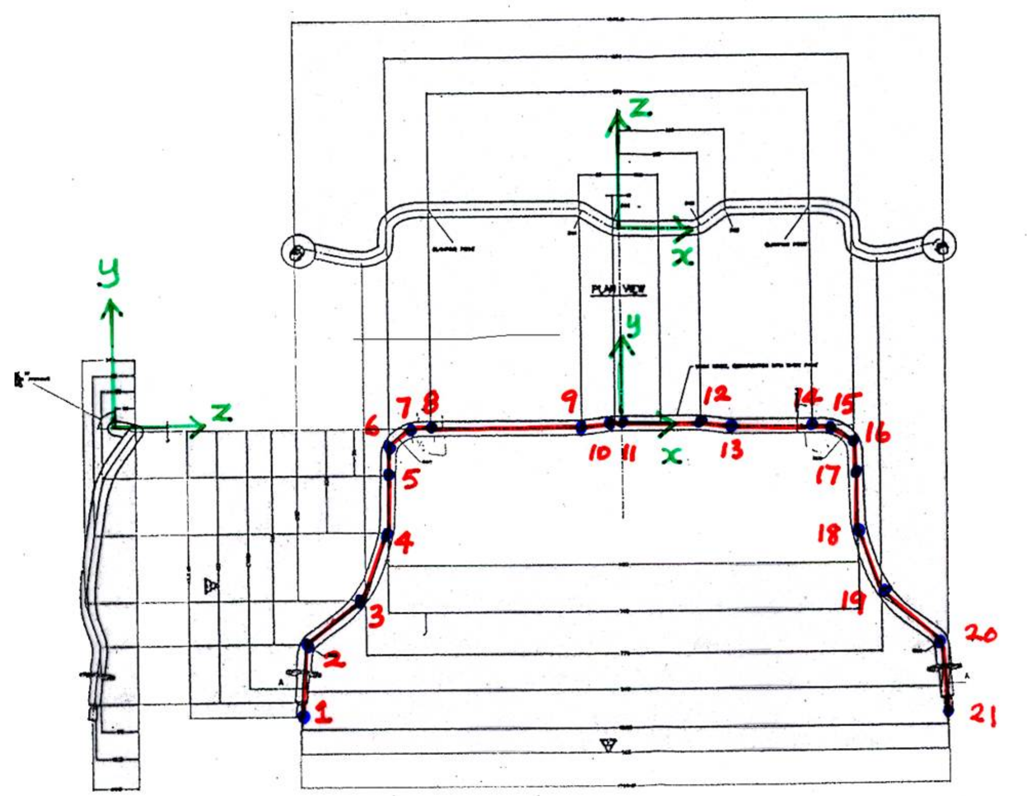
$\mathrm{K} = \mathrm{L}^{\mathrm{T}}\mathrm{kL}$

where L is the local-global transformation matrix. After applying boundary conditions, the system equation is given by:

$\mathrm{KQ} = \mathrm{F}$

where Qis the displacement and F is load.

3-Element Geometry:



Z
X
PLAN VIEW
y
y
Z
x
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21

Node	x	y	z
1	-0.4846	-0.4386	-0.0355
2	-0.4745	-0.329	-0.025
3	-0.391	-0.263	-0.047
4	-0.3515	-0.162	-0.033
5	-0.349	-0.076	-0.02
6	-0.347	-0.056	0
7	-0.297	-0.006	0.03
8	-0.285	-0.006	0.03
9	-0.06	-0.006	0.03
10	-0.01	0	0
11	0	0	0
12	0.12	0	0
13	0.16	-0.006	0.03
14	0.285	-0.006	0.03
15	0.297	-0.006	0.03
16	0.347	-0.056	0
17	.0349	-0.076	-0.02
18	0.3515	-0.162	-0.033
19	0.391	-0.263	-0.047
20	0.4745	-0.329	-0.025
21	0.4846	-0.4386	-0.0355

4-Geometry Parameters

Radius,

Area,

Moment of inertia ,

Polar moment of inertia,

5-Material Parameters:

Material: Carbonized Steel Modulus of elasticity: 207 GPa

Modulus of rigidity : 79.3 Gpa

6-Boundary Conditions

There are two clamping points which are at node 8 and node 14 which give boundary conditions as follows:

	Node 8	Node 14
x-displacement	0	0
y-displacement	0	0
z-displacement	0	0
x-rotation	0	0

7-Load

Point load at the end of the bar is as below:

i. Compression at both ends.

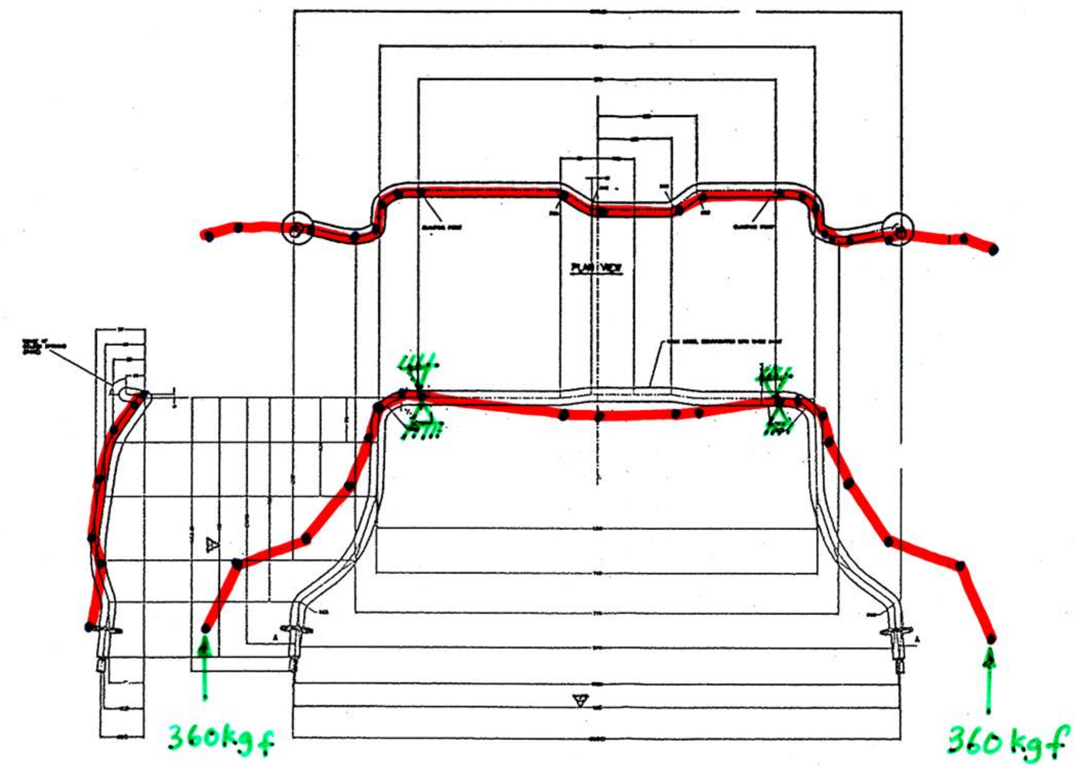
ii. Tension at both ends.

iii. Compression at one end and tension at the other.

Each end is applied 360 kgf of point load.

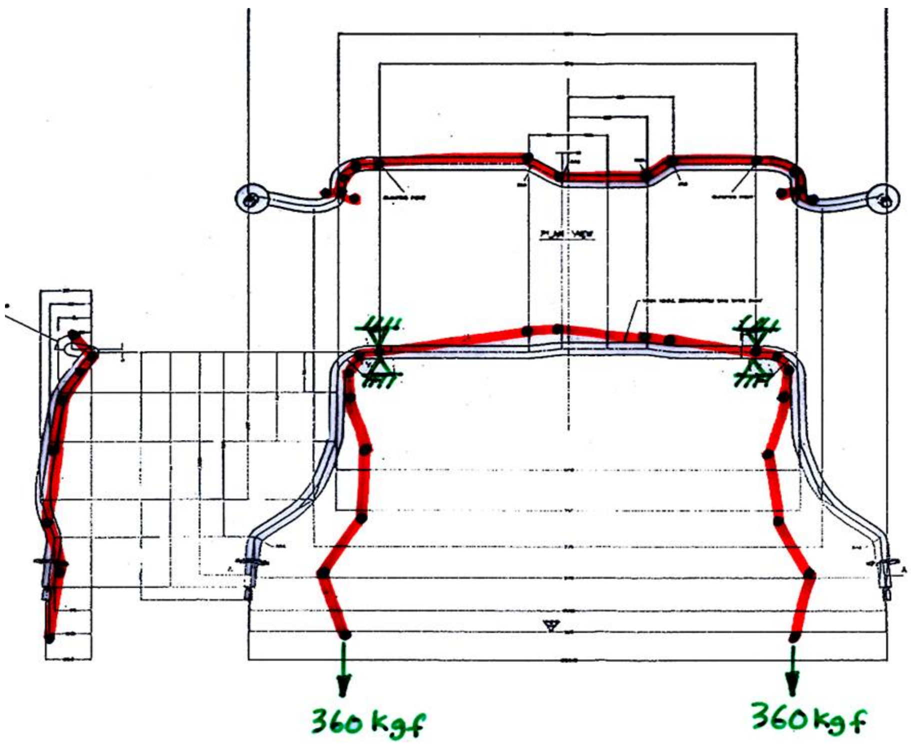
8-Result

i. Deflection of the stabilizer bar with both ends under compression.



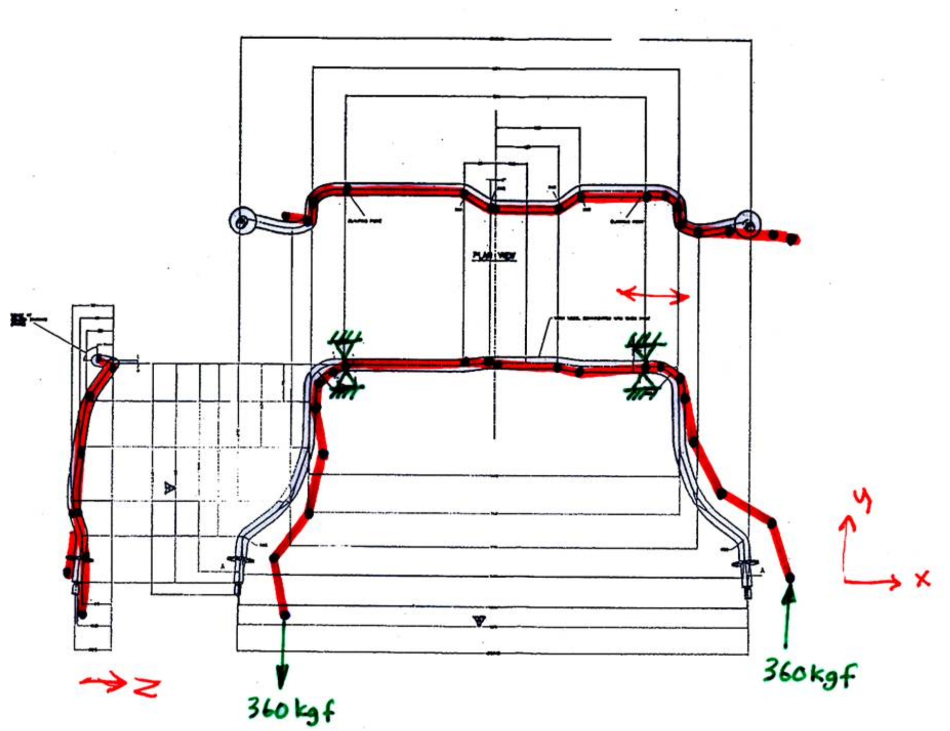
Node	x-deflection	y-deflection	z-deflection
1	-115.28	52.412	-7.2
11	-0.15	-23.97	-1.3
21	116.257	52.861	-7.2

ii. Deflection of the stabilizer bar with both ends under tension.



Node	x-deflection	y-deflection	z-deflection
1	115.2	-52.41	7.195
11	0.1519	23.97	1.304
21	-116.2	-52.86	7.195

iii. Deflection of the stabilizer bar with compression at one end and tension at the other end.



Node	x-deflection	y-deflection	z-deflection
1	67.145	-30.20	7.195
11	0.107	-0.12	-0.24
21	68.12	30.65	-7.184

9- Appendix Input data for Frame3D program for cases where both ends are under compression.

<<3-D Frame Analysis >>
Stabilizer Bar Deflection Analysis

NN	NE	NM	NDIM	NEN	NDN	NNREF
21	20	1	3	2	6	3

ND	NL	NMPC
10	2	0

Node#	X	Y	Z
1	-0.4846	-0.4386	-0.0355
2	-0.4745	-0.329	-0.025
3	-0.391	-0.263	-0.047
4	-0.3515	-0.162	-0.033
5	-0.349	-0.076	-0.02
6	-0.347	-0.056	0
7	-0.297	-0.006	0.03
8	-0.285	-0.006	0.03
9	-0.06	-0.006	0.03
10	-0.01	0	0
11	0	0	0
12	0.12	0	0
13	0.16	-0.006	0.03
14	0.285	-0.006	0.03
15	0.297	-0.006	0.03
16	0.347	-0.056	0
17	0.349	-0.076	-0.02
18	0.3515	-0.162	-0.033
19	0.391	-0.263	-0.047
20	0.4745	-0.329	-0.025
21	0.4846	-0.4386	-0.0355
22	-0.347	-0.006	0.03
23	0	10	0
24	0.347	-0.006	0.03

Elem#	N1	N2	Ref_Pt	Mat#	Area	Iy	Iz	J	UDLy'	UDLz'
1	1	2	22	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
2	2	3	22	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
3	3	4	22	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
4	4	5	22	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
5	5	6	22	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
6	6	7	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
7	7	8	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
8	8	9	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
9	9	10	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
10	10	11	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
11	11	12	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
12	12	13	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
13	13	14	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
14	14	15	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
15	15	16	23	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
16	16	17	24	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
17	17	18	24	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
18	18	19	24	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
19	19	20	24	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0
20	20	21	24	1	2.7465E-04	6.0026E-09	6.0026E-09	1.2005E-08	0	0

DOF#	Displ.
43	0
44	0
45	0
46	0
47	0
79	0
80	0
81	0
82	0
83	0

DOF#	Load
2	3531.6
122	3531.6

MAT#	E	G
1	2.07E+11	7.93E+10

B1	i	B2	j	B3	(Multi-point constr. B1*Qi+B2*Qj=B3)

For the case where both ends are under tension, the load on the degree of freedom 2 and 122 becomes -3531.6 and -3531.6, while for the case where one end are under tension and the other are under compression, the load becomes -3531.5 and 3531.6.