4

OPTIMISATION

- Introduction
- Single Variable Unconstrained Optimisation
- Multivariable Unconstrained Optimisation
- Linear Programming

4.1 Introduction

- In an engineering analysis, sometimes *extremities*, either *minimum* or *maximum* value, has to be obtained.

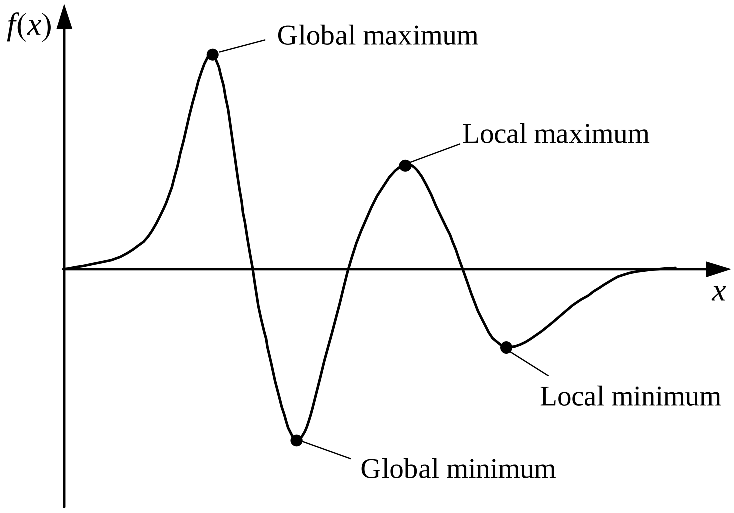


FIGURE 4.1 Extremities for a single variable function

- Extremity value can be obtained via *optimisation*, which is divided into:

1. Unconstrained optimisation — $f'(x) = 0$.
2. Constrained optimisation — linear/non-linear programming.

4.2 Single Variable Unconstrained Optimisation

- Extremities, if any, can be evaluated using either the quadratic interpolation method or the Newton method using the condition of $f'(x)=0$.

- For the **quadratic interpolation method**, consider a second order Lagrange interpolation equation as followed:

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \tag{4.1}$$

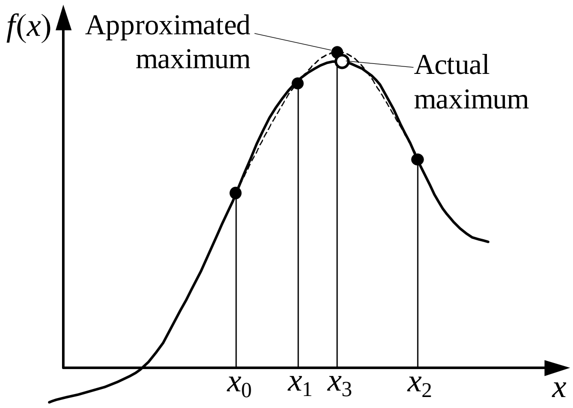


FIGURE 4.2 Evaluation of extremities using a quadratic function

Eq. (4.1) is differentiated to yield:

$$f'(x)=0=\frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Thus, it can be rearranged to get an optimised value of $x=x_3$:

$$x_3=\frac{f(x_0)(x_1^2-x_2^2)+f(x_1)(x_2^2-x_0^2)+f(x_2)(x_0^2-x_1^2)}{2f(x_0)(x_1-x_2)+2f(x_1)(x_2-x_0)+2f(x_2)(x_0-x_1)} \tag{4.2}$$

Eq. (4.2) can be repeated until converged.

Example 4.1

Use the quadratic interpolation method to obtain a maximum value of the following function accurate to four decimal places:

$$f(x) = \sin x - 0.2x^2$$

using initial values of $x_0 = 0$, $x_1 = 1$ dan $x_2 = 2$.

Solution

From the given function:

$$f(0) = \sin(0) - 0.2(0)^2 = 0$$
$$f(1) = \sin(1) - 0.2(1)^2 = 0.6415$$
$$f(2) = \sin(2) - 0.2(2)^2 = 0.1093$$

Using Eq. (4.2), the value of x_3 can be estimated as followed:

$$x_3 = \frac{(0)(1^2 - 2^2) + (0.6415)(2^2 - 0^2) + (0.1093)(0^2 - 1^2)}{2(0)(1-2) + 2(0.6415)(2-0) + 2(0.1093)(0-1)} = 1.0466$$
$$f(x_3) = \sin(1.0466) - 0.2(1.0466)^2 = 0.6466$$

The overal process is as followed:

i	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$
1	0	0	1	0.6415	2	0.1093	1.0466	0.6466
2	1	0.6415	1.0466	0.6466	2	0.1093	1.1057	0.6493
3	1	0.6415	1.0466	0.6466	1.1057	0.6493	1.1110	0.6493
4	1.0466	0.6466	1.1057	0.6493	1.1110	0.6493	1.1105	0.6493
5	1.1057	0.6493	1.1105	0.6493	1.1110	0.6493	1.1105	0.6493

Hence, the maximum value is $f(x) = 0.6493$ at $x = 1.1105$.

- An extremity can either be a minimum or maximum value, or otherwise, depending on the second derivative $f''(x)$:

1. $f''(x) > 0$ — $f(x)$ is minimum,
2. $f''(x) < 0$ — $f(x)$ is maximum,
3. $f''(x) = 0$ — the coordinate $[x, f(x)]$ is an *inflection* point.

- For the **Newton method**, consider an equation similar to the Newton-Raphson formula (requiring only one initial value):

$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}$$

If $f(x)$ is the first derivative of $g(x)$, i.e. $g(x) = f'(x) = 0$, this the root of $g(x)$ is an extremity for $f(x)$, or

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \tag{4.3}$$

Example 4.2

Use the Newton method to obtain the maximum value of the function:

$$f(x) = \sin x - 0.2x^2$$

using an initial value of $x_0 = 1$. Use the convergence criterion of an approximated error of less than 0.05%.

Solution

From the given function:

$$f'(x) = \cos x - 0.4x$$
$$f''(x) = -\sin x - 0.4$$

Using Eq. (4.3), the iteration formula is:

$$x_{i+1} = x_i + \frac{\cos x_i - 0.4x_i}{\sin x_i + 0.4}$$

which produces

i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	$\|\varepsilon_a\|$ (%)
0	1	0.64147	0.140302	–1.24147	11.30
1	1.11301	0.64928	–0.00324	–1.29703	0.250
2	1.11051	0.64928	–1.4E–06	–1.29593	0.000
3	1.11051	0.64928	–2.5E–13	–1.29593	0.000

Hence, the maximum value is $f(x) = 0.64928$ at $x = 1.11051$.

4.3 Multivariable Unconstrained Optimisation

- For a multivariable case, extremities can be evaluated using the *gradient method* via the *steepest slope* condition.

- For a multivariable case, the gradient vector can be written as

$$\nabla f = \left(\frac{\partial}{\partial x_1} f(\mathbf{x}), \frac{\partial}{\partial x_2} f(\mathbf{x}), \ldots, \frac{\partial}{\partial x_n} f(\mathbf{x}) \right)^{\mathrm{T}}$$

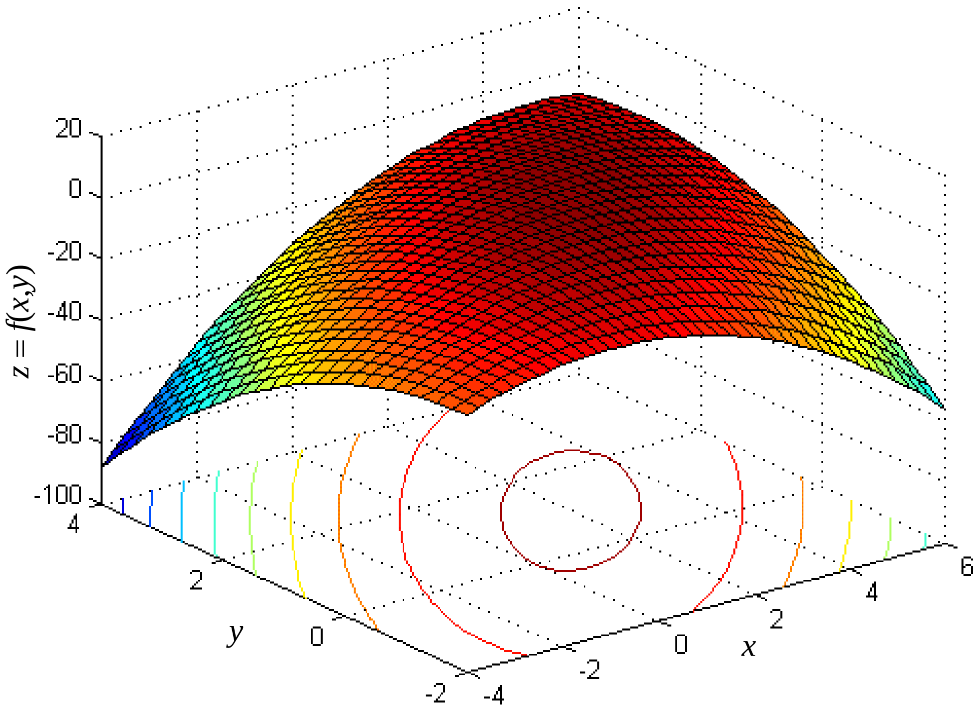


FIGURE 4.3 Optimisation for the 2-D case $z = f(x, y)$

- Consider the equation of two variables:

$$z = f(x, y) \tag{4.4}$$

The objective is to obtain a condition where $\nabla f = \mathbf{0}$, and for this case:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

This vector will guide the solution towards a normal direction (or *orthogonal*) to a contour line of constant $f(x,y)$.

If h is the distance needed to reach the extremity, the next approximation to x and y are

$$x = x_0 + \frac{\partial f}{\partial x} h \quad , \quad y = y_0 + \frac{\partial f}{\partial y} h \tag{4.5}$$

Thus a function $g(h)$ can be formed such that

$$g(h) = f\left(x_0 + \frac{\partial f}{\partial x} h, y_0 + \frac{\partial f}{\partial y} h \right) \tag{4.6}$$

and the relation $g'(h) = 0$ gives the optimised h and hence the optimised values of x and y.

- For the multivariable cases, the type of extremities is determined using the *Hessian* $|H|$ parameter, which has been defined as

$$|H| = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \tag{4.7}$$

The parameter $|H|$ is equivalent to $f''(x)$ for a single variable case, where:

1. $|H| > 0$ and $\partial^2 f / \partial x^2 > 0$ — $f(x,y)$ has a local minimum,
2. $|H| > 0$ dan $\partial^2 f / \partial x^2 < 0$ — $f(x,y)$ has a local maximum,
3. $|H| < 0$ — $f(x,y)$ has a *plateau*.

Example 4.3

Maximise the following function:

$$f(x, y) = 2xy + 2x - x^2 - 2y^2$$

using the gradient method of the steepest slope using an initial values of $x_0 = -1$ dan $y_0 = 1$. Get the answer accurate to three decimal places.

Solution

In the first iteration:

$$\frac{\partial f}{\partial x} = 2y + 2 - 2x = 2(1) + 2 - 2(-1) = 6$$

$$\frac{\partial f}{\partial y} = 2x - 4y = 2(-1) - 4(1) = -6$$

$$
\begin{aligned}
g(h) &= f(-1+6h, 1-6h) \\
&= 2(-1+6h)(1-6h) + 2(-1+6h) - (-1+6h)^2 - 2(1-6h)^2 \\
&= -180h^2 + 72h - 7
\end{aligned}
$$

$$g'(h) = 0 = -360h + 72 \quad \Rightarrow \quad h = 0.2$$

Thus after the first iteration:

$$x = -1 + 6(0.2) = 0.2$$
$$y = 1 - 6(0.2) = -0.2$$

In the second iteration:

$$\frac{\partial f}{\partial x} = 2(-0.2) + 2 - 2(0.2) = 1.2$$
$$\frac{\partial f}{\partial y} = 2(0.2) - 4(-0.2) = 1.2$$

$$g(h) = f(0.2 + 1.2h, -0.2 + 1.2h) = -1.44h^2 + 2.88h + 0.2$$
$$g'(h) = 0 = -2.88h + 2.88 \quad \Rightarrow \quad h = 1$$
$$x = 0.2 + 1.2(1) = 1.4$$
$$y = -0.2 + 1.2(1) = 1$$

The overall process is as followed:

i	x_{i-1}	y_{i-1}	$\partial f / \partial x$	$\partial f / \partial y$	h	x_i	y_i
1	–1	1	6	–6	0.2	0.2	–0.2
2	0.2	–0.2	1.2	1.2	1	1.4	1
3	1.4	1	1.2	–1.2	0.2	1.64	0.76
4	1.64	0.76	0.24	0.24	1	1.88	1
5	1.88	1	0.24	–0.24	0.2	1.928	0.952
6	1.928	0.952	0.048	0.048	1	1.976	1
7	1.976	1	0.048	–0.048	0.2	1.986	0.990
8	1.986	0.990	0.0096	0.0096	1	1.995	1
9	1.995	1	0.0096	–0.0096	0.2	1.997	0.998
10	1.997	0.998	0.00192	0.00192	1	1.999	1
11	1.999	1	0.00192	–0.00192	0.2	1.999	1.000
12	1.999	1.000	0.00038	0.00038	1	2.000	1
13	2.000	1	0.00038	–0.00038	0.2	2.000	1.000

Finally, the solution converges at the 13-th iteration where $x = 2$ dan $y = 1$ resulting in a maximum value of $f(x, y) = f(2,1) = 2$.

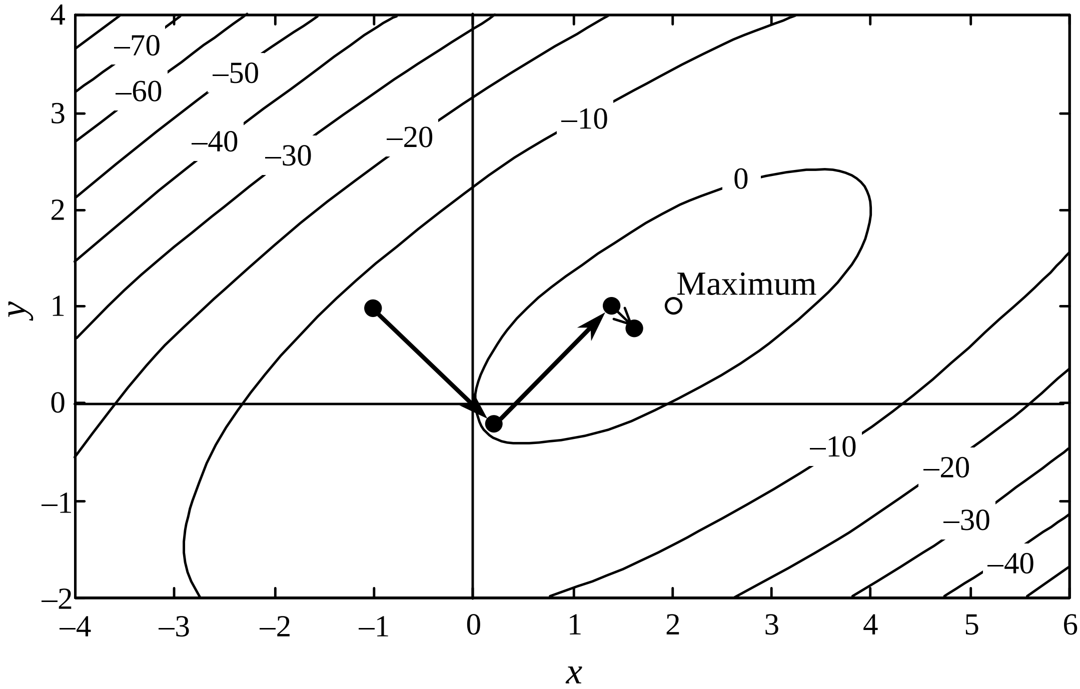


FIGURE 4.4 Propagation of estimated points of Example 4.3

4.4 Linear Programming

- In this topic, only the linear case is considered.
- The objective of linear programming is to *minimise* or *maximise* an objective function Z, i.e.,

  Maksimumkan: $$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{4.8}$$

  Eq. (4.8) is subjected to several *constraints*, i.e.

  $$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i \tag{4.9}$$

  If the variable x_j represents a positive physical parameter, thus

  $$x_j \ge 0 \tag{4.10}$$

- The simplest approach is via a **graphical method**.

Example 4.4

Use the graphical method to maximise the following objective function:

$$Z = 150x + 175y$$

where the conditions or constraints are:

(1) $7x + 11y \le 77$,
(2) $10x + 8y \le 80$,
(3) $x \le 9$,
(4) $y \le 6$,
(5) $x \ge 0$,
(6) $y \ge 0$.

Solution

From the figure, the optimum point is $\left(4\tfrac{8}{9}, 3\tfrac{8}{9}\right)$ which produces the maximum value of $Z = 1413\tfrac{8}{9}$. Noted that condition (3) is redundant.

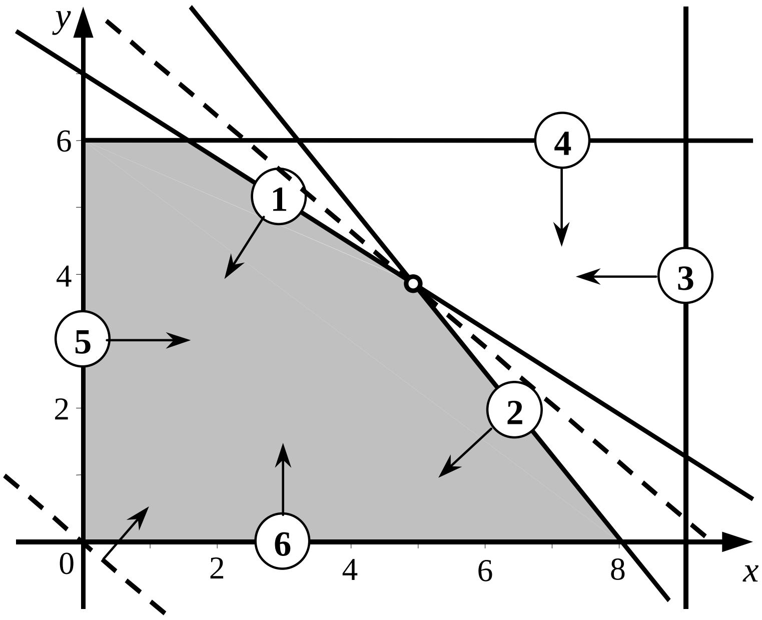


FIGURE 4.5 Linear programming graph for Example 4.4

- One of the numerical approach is the **simplex method**, where the searching for the optimum point is guided by the *slag variable* S_i, as followed:

$$Z - c_1 x_1 - c_2 x_2 - \cdots - c_n x_n = 0 \tag{4.11}$$

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n + S_i = b_i \tag{4.12}$$

$$x_j + S_j = 0 \tag{4.13}$$

If this system contains k equations and l variables including the slag variables, where usually $k < l$, hence there are (l–k) variables which has to be made zeros (*non-basis* — a non-zero variable is known as *basis*).

The Gauss-Jourdan elimination can be performed to minimise the objective function.

The elimination can be stopped when all the basis variables become zeros.

Example 4.5

Repeat Example 4.4 using the simplex method.

Solution

The system can be rewritten as followed:

Maximise: $Z - 150x - 175y = 0$,

With conditions:
(1) $7x + 11y + S_1 = 77$,
(2) $10x + 8y + S_2 = 80$,
(3) $x + S_3 = 9$,
(4) $y + S_4 = 6$,
(5) $x, y, S_1, S_2, S_3, S_4 \geq 0$.

Begin with $Z = x = y = 0$. Then form the following table:

Basis	Z	x	y	S_1	S_2	S_3	S_4	Solution
Z	1	-150	-175	0	0	0	0	0
S_1	0	7	11	1	0	0	0	77
S_2	0	10	8	0	1	0	0	80
S_3	0	1	0	0	0	1	0	9
S_4	0	0	1	0	0	0	1	6

At column x, the element at row S_2 can be a pivot, hence x is selected to be the inbound variable replacing S_2. Then, perform the Gauss elimination:

Basis	Z	x	y	S_1	S_2	S_3	S_4	Solution
Z	1	0	-55	0	15	0	0	1200
x	0	1	0.8	0	0.1	0	0	8
S_1	0	0	5.4	1	-0.7	0	0	21
S_3	0	0	-0.8	0	-0.1	1	0	1
S_4	0	0	1	0	0	0	1	6

The coefficient of y at row Z is still negative, thus Z is still not maximum. Hence, y is selected to replace S_1:

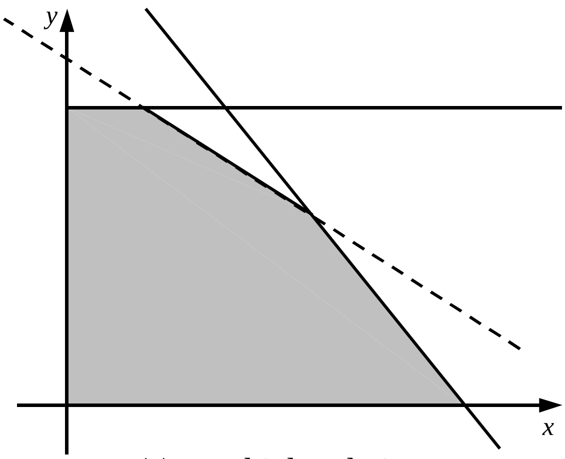
Basis	Z	x	y	S_1	S_2	S_3	S_4	Solution
Z	1	0	0	10.1852	7.8704	0	0	1413.889
x	0	1	0	-0.1481	0.2037	0	0	4.889
y	0	0	1	0.1852	-0.1296	0	0	3.889
S_3	0	0	0	0.1481	-0.2037	1	0	4.111
S_4	0	0	0	-0.1852	0.1296	0	1	2.111

Therefore the maximum of Z is 1413.889 which is produced at $x = 3.889$ and $y = 4.889$.

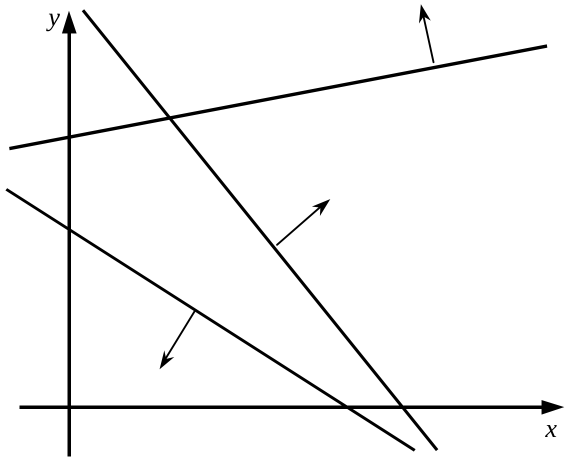
- In linear and non-linear programming, there are four possible outcomes:

  1. Unique solution,
  2. Multiple solutions,
  3. No possible solution,
  4. Unbounded problem.

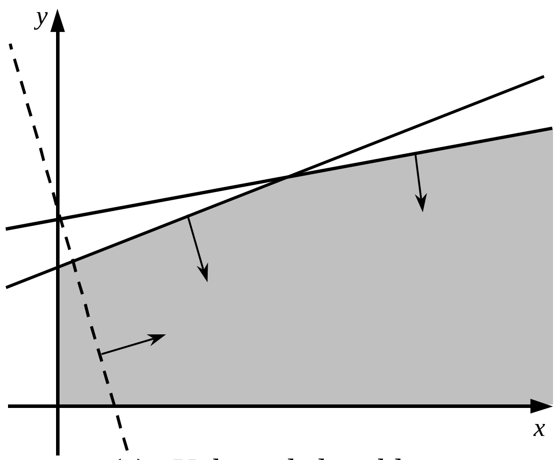
  For cases 2-4, the simplex method cannot be used.



(a) Multiple solutions



(b) No possible solution



(c) Unbounded problem

FIGURE 4.6 Cases where the simplex method is not applicable

Exercises

1. Obtain the minimum value of the following function at $x \geq 0$ using the quadratic interpolation function using the initial values of 0.1, 0.5 and 5.0, and the Newton method using the initial value of 0.5:

$$f(x) = x + \frac{1}{x}$$

2. Obtain the maximum value of the following function via the steepest slope with the initial value of $(x, y) = (0, 0)$:

$$f(x) = 3.5x + x^2 - x^4 - 2xy + 2y - y^2$$

3. A company produces two types of products, A and B. These products are produced during normal working days of 40 hours per week and are marketed on the same weekends. The company needs 20 kg and 5 kg of raw materials for products A and B, respectively. However, the company warehouse can only stores 10,000 kg of raw materials per week. Only one product is produced at one time, where product A requires 0.05 hour, while product B requires 0.15 hour. Nevertheless, the temporary storage section can only keep 550 products per week. Product A is sold at RM45 per unit while product B is sold at RM30 per unit. By using the linear programming using the simplex method:

   a. Maximise the company profit.
   b. Which factor where its increase leads to the fastest increase in profit: raw materials, capacity of temporary storage section or production time?