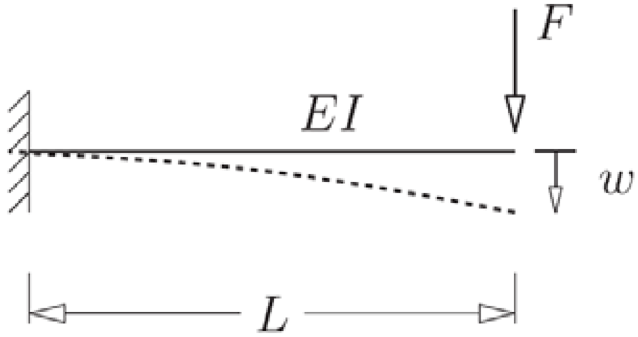
1 *Static analysis*



$$w(x) = -\frac{F}{EI}\frac{x^3}{6} + \frac{FL}{EI}\frac{x^2}{2}$$

The vertical end deflection under the load is then given by

$$w = \frac{FL^3}{3EI}$$

Assume L=1 and that the load F and the bending stiffness EI are random variables with mean values of 1 and standard deviations of 0.1. What are the mean value and the standard deviation of w?

One can attempt to compute the mean value of w by inserting the mean values of F and EI into the above equation.

This results in $\overline{w}$ = 1/3

Alternately, we might try to solve the problem by Monte-Carlo simulation, i.e. by generating random numbers representing samples for F and EI, compute the deflection for each sample and estimate the statistics of w from those values.

```
M=1000000;
F=1+.1*randn(M,1);
EI=1+.1*randn(M,1);
w=F./EI/3.;
wm=mean(w)
ws=std(w)
cov=ws/wm
```

Running this script three times, we obtain

```
wm=0.33676
ws=0.048502
```

```
cov=0.14403

wm=0.33673
ws=0.048488
cov=0.14399

wm=0.33679
ws=0.048569
cov=0.14421
```

In these results,

`wm` denotes the mean value,
`ws` the standard deviation, and
`cov` the coefficient of variation (the standard deviation divided by the mean).

It can be seen that the mean value is somewhat larger than 1/3 .

Also, the coefficient of variation of the deflection is considerably larger than the coefficient of variation of either F or EI.

Exercise 1
Consider a cantilever beam as discussed in the example above, but now with a varying bending stiffness;

$$EI(x) = \frac{EI_0}{1-\frac{x}{2L}}.$$

a) for deterministic values of F, L and EI_0

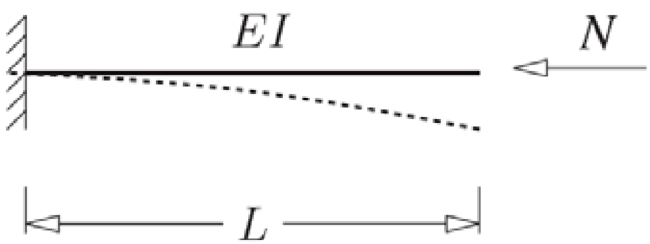
b) for random values of F, L and EI_0. Assume that these variables have a mean value of 1 and a standard deviation of 0.05. Compute the mean value and the standard deviation of the end deflection using Monte Carlo simulation.

Solution:
The deterministic end deflection is

$$w_d = \frac{5FL^3}{12EI_0}.$$

A Monte Carlo simulation with 1000000 samples yields a mean value of `wm`=0.421 and a standard deviation of `ws`=0.070.

2 *Buckling analysis*



$$N_{cr} = \lambda_1^2 EI = \frac{\pi^2 EI}{4L^2}$$

The magnitude of the corresponding deflection remains undetermined.

Now assume that $L = 1$ and the load N is a Gaussian random variable with a mean value of 2 and standard deviation of 0.2, and the bending stiffness EI is a Gaussian random variable with a mean value of 1 and standard deviation of 0.1.

What is the probability that the actual load N is larger than the critical load N_{cr}?

```
M=1000000;
N3=2+.2*randn(M,1);
EI=1+.1*randn(M,1);
Ncr=pi^2*EI/4.;
indicator = N>Ncr;
pf=mean(indicator)
```

```
pf = 0.070543
pf = 0.070638
pf = 0.070834
```

In these results, `pf` denotes the mean value of the estimated probability.
This problem has an exact solution which can be computed analytically: `pf=`0.0705673.

Exercise 2
Consider the same stability problem as above, but now assume that the random variables involved are N, L and EI_0.

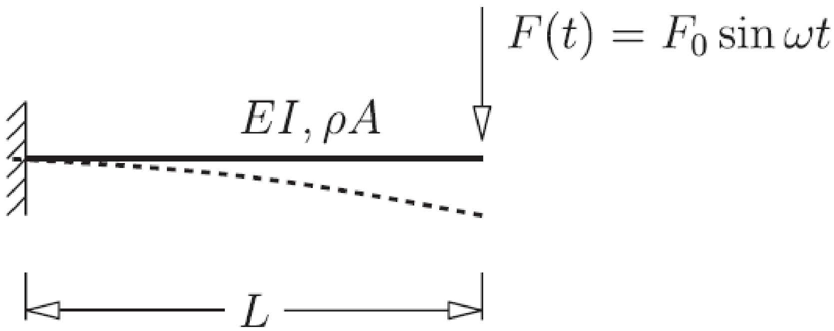
Presume that these variables have a mean value of 1 and a standard deviation of 0.05.

Compute the mean value and the standard deviation of the critical load applying Monte Carlo simulation using one million samples. Compute the probability that the critical load is less than 2.

Solution: Monte Carlo simulation results in `mn`=2.4675, `sn`=0.12324 and `pf`=9.3000e-05.

The last result is not very stable, i.e. it varies quite considerably in different runs.

3 ***Dynamic analysis***



Now, we consider the same simple cantilever under a dynamic loading *F*(*t*).

For this beam with constant density ρ, cross sectional area *A* and bending stiffness *EI* under distributed transverse loading $p(x, t)$, the dynamic equation of motion is

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(x, t)$$

We would like to compute the probability that the load as given is close to a resonance situation, i.e. the ratio of the excitation frequency ω and the first natural frequency ω_1 of the system is close to 1.

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

The fundamental natural circular frequency ω_1 is

$$\omega^2 = \frac{\lambda_1^4 EI}{\rho A} = \frac{12.362 EI}{\rho A L^4}$$

Now we assume that the excitation frequency ω is a random variable with a mean value of 0.3 and a standard deviation of 0.03.

The bending stiffness is a random variable with mean value 0.1 and standard deviation 0.01, the cross sectional area is random with a mean value of 1 and a standard deviation of 0.05.

The density is deterministic ρ=1, so is the length L=1.

We want to compute the probability that the ratio $\omega/\omega1$ lies between 0.99 and 1.01.

```
M=1000000;
om=1+0.1*randn(M,1);
EI=0.1+0.01*randn(M,1);
A=1+0.05*randn(M,1);
om1=sqrt(EI./A*12.362);
ind1 = om./om1>0.99;
ind2 = om./om1<1.01;
indicator = ind1.*ind2;
pr=mean(indicator)
```

Running

```
pr = 0.046719
pr = 0.046946
pr = 0.046766
```

In these results, `pr` denotes the mean value of the estimated probability.

Exercise 3 (Dynamic deflection)

Now assume that the random variables involved in the above example are A, L and EI.

Let these variables have a mean value of 1 and a standard deviation of 0.05.

Compute the mean value and the standard deviation of the fundamental natural circular frequency ω_1 using Monte Carlo simulation with one million samples.

Compute the probability that ω_1 is between 2 and 2.5.

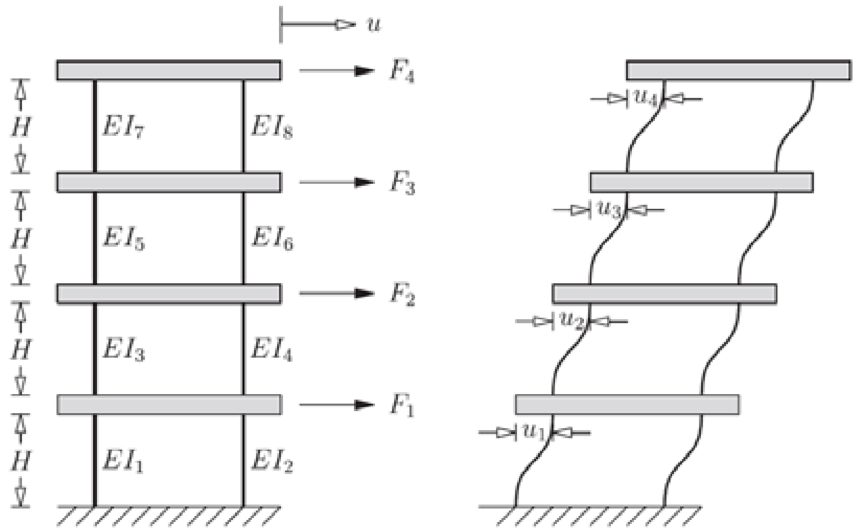
Solution:
Monte Carlo simulation results in the mean value `mo`=3.55, the standard deviation `so`=0.38 and the probability is of the order of `pf`=2.8e-4.

4 *Structural analysis*

A four-story stucture is subjected to four static loads F_i, i = 1, 2, 3, 4. The floor slabs are assumed to be rigid and the columns have identical length H =4m and different bending stiffnesses EI_k, k=1 . . . 8.

Loads and stiffnesses are random variables. The loads are normally distributed with a mean value of 20kN and a COV of 0.4, the stiffnesses are normally distributed with a mean value of 10MNm^2 and a COV of 0.2.

All variables are pairwise independent.



compute

- the mean value and standard deviation as well as the coefficient of variation of the horizontal displacement *u* of the top story,
- the probability p_F that *u* exceeds a value of 0.1 m.

The analysis is to be based on linear elastic behavior of the structure excluding effects of gravity.

The top story deflection can be calculated by adding the interstory relative displacements

$$u_4 = \frac{F_4 H^3}{12(EI_7 + EI_8)}$$

$$u_3 = \frac{(F_3 + F_4)H^3}{12(EI_5 + EI_6)}$$

$$u_2 = \frac{(F_2 + F_3 + F_4)H^3}{12(EI_3 + EI_4)}$$

$$u_1 = \frac{(F_1 + F_2 + F_3 + F_4)H^3}{12(EI_1 + EI_2)}$$

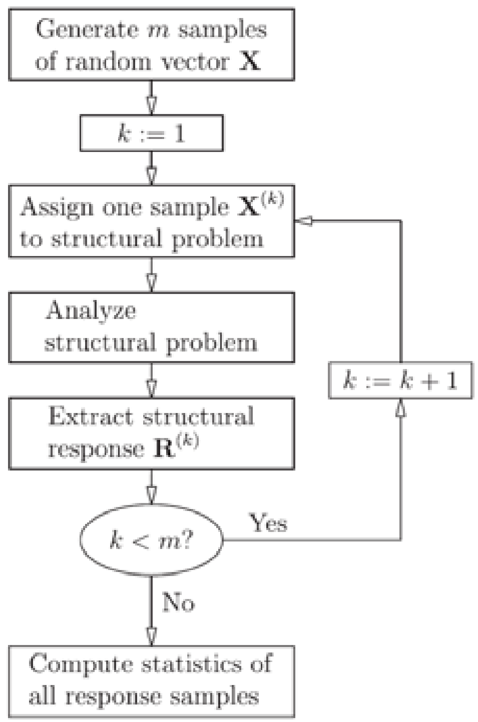
$$u = u_1 + u_2 + u_3 + u_4$$

```
Fbar=20;
sigmaF = Fbar*0.4;
EIbar=10000;
sigmaEI = EIbar*0.2;

NSIM=1000000;
ULIM=0.1;
UU=zeros(NSIM,1);

F1=Fbar + sigmaF*randn(NSIM,1);
F2=Fbar + sigmaF*randn(NSIM,1);
F3=Fbar + sigmaF*randn(NSIM,1);
F4=Fbar + sigmaF*randn(NSIM,1);

EI1 = EIbar + sigmaEI*randn(NSIM,1);
EI2 = EIbar + sigmaEI*randn(NSIM,1);
EI3 = EIbar + sigmaEI*randn(NSIM,1);
EI4 = EIbar + sigmaEI*randn(NSIM,1);
EI5 = EIbar + sigmaEI*randn(NSIM,1);
EI6 = EIbar + sigmaEI*randn(NSIM,1);
EI7 = EIbar + sigmaEI*randn(NSIM,1);
EI8 = EIbar + sigmaEI*randn(NSIM,1);

H=4;

u4=F4./(EI7+EI8)/12*H^3;
u3=(F3+F4)./(EI5+EI6)/12*H^3;
u2=(F2+F3+F4)./(EI3+EI4)/12*H^3;
u1=(F1+F2+F3+F4)./(EI1+EI2)/12*H^3;
u=u1+u2+u3+u4;

UM=mean(u)
US=std(u)
COV=US/UM
indic=u>ULIM;
PF=mean(indic)
```

```
UM = 0.054483
US = 0.012792
COV = 0.23478
PF = 7.1500e-04
```

5 Monte Carlo simulation

This is a frequently used method to deal with the effect of random uncertainties. Typically its application aims at integrations such as the computation of expected values (e.g. mean or standard deviation).



In order to illustrate the close relationship between the computation of probabilities and integration, consider the determination of the area of a quarter circle of unit radius.

As we know, the area is $\pi/4$, which can be computed using analytical integration.

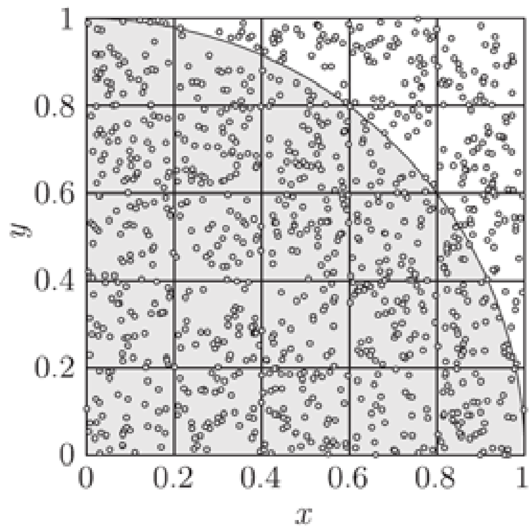
Using the Monte Carlo Method we can obtain approximations to this result based on elementary function evaluations.

When we use 1000 uniformly distributed random numbers x and y, and count the number N_c of pairs (x, y) for which $x^2 + y^2 < 1$, we get an estimate $\pi/4 \approx Nc/1000 = 791/1000 = 0.791$.

This differs from the exact result $\pi/4 = 0.7854$ by about 1%.

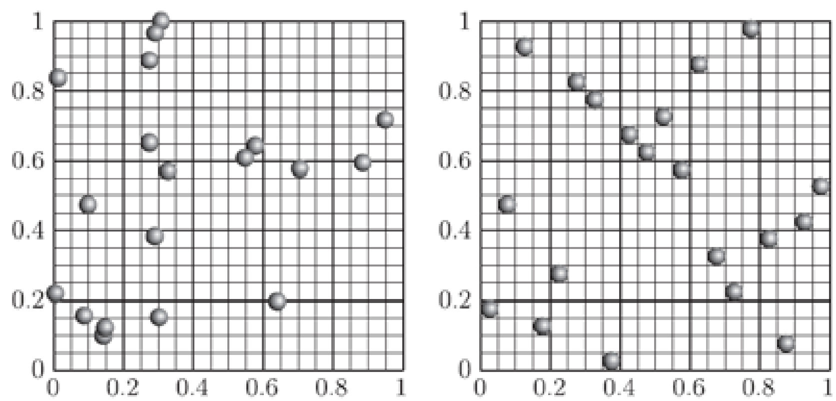
```
M=1000
x=rand(M,1);
y=rand(M,1);
r2=x.^2+y.^2;
indic = r2<1;
NC=sum(indic)
```

```
area=NC/M
fid=fopen('circle.txt','w');
for i=1:size(x)
fprintf(fid, '%g %g\n', x(i), y(i));
end
fclose(fid);
```



6 Latin Hypercube sampling

In order to reduce the statistical uncertainty associated with Monte Carlo estimation of expected values, alternative methods have been developed. One such strategy is the Latin Hypercube sampling method (Imam and Conover 1982).



The coefficient of correlation in the left figure is ρ=0.272, whereas the coefficient of correlation in the right figure is only ρ=−0.008, i.e., it is virtually zero.