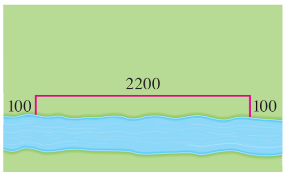
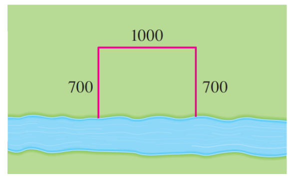
Optimization Problems

EXAMPLE 1: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

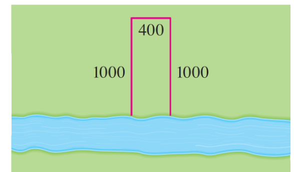
Solution: Note that the area of the field depends on its dimensions:



Area = 100 · 2200 = 220,000 ft²

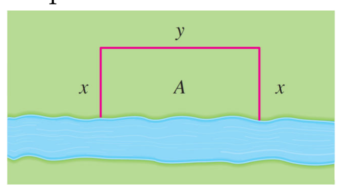


Area = 700 · 1000 = 700,000 ft²



Area = 1000 · 400 = 400,000 ft²

To solve the problem, we first draw a picture that illustrates the general case:



The next step is to create a corresponding mathematical model:

$$\text{Maximize: } A = xy$$

$$\text{Constraint: } 2x + y = 2400$$

We now solve the second equation for y and substitute the result into the first equation to express A as a function of one variable:

$$2x + y = 2400 \quad \Longrightarrow \quad y = 2400 - 2x \quad \Longrightarrow \quad A = xy = x(2400 - 2x) = 2400x - 2x^2$$

To find the absolute maximum value of $A = 2400x - 2x^2$, we use

THE CLOSED INTERVAL METHOD: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

We first show that $0 \le x \le 1200$. Indeed,

$$y \ge 0 \quad \Longrightarrow \quad 2400 - 2x \ge 0 \quad \Longrightarrow \quad 2400 \ge 2x \quad \Longrightarrow \quad 1200 \ge x$$

On the other hand, $x \ge 0$. Combining these two inequalities gives $0 \le x \le 1200$. The derivative of $A(x)$ is

$$A'(x) = (2400x - 2x^2)' = 2400x' - 2(x^2)' = 2400 \cdot 1 - 2 \cdot 2x = 2400 - 4x$$

so to find the critical numbers we solve the equation

$$2400 - 4x = 0 \quad \Longrightarrow \quad 2400 = 4x \quad \Longrightarrow \quad x = \frac{2400}{4} = 600$$

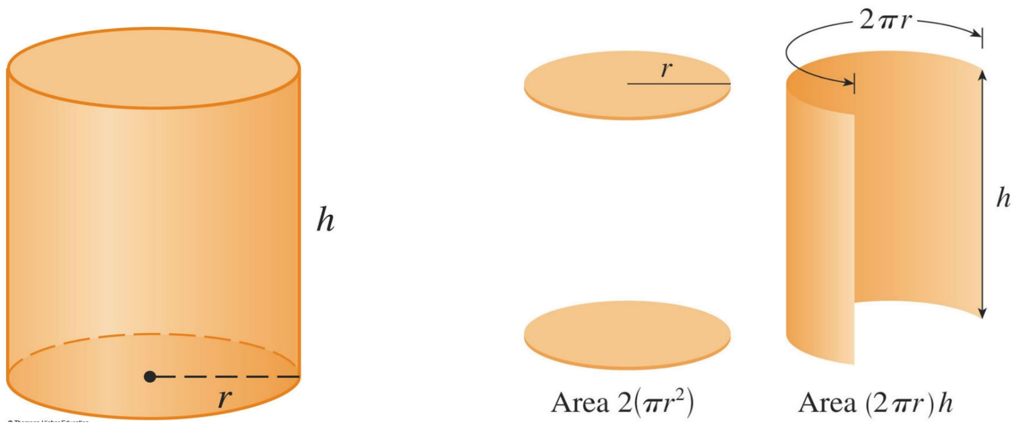
To find the maximum value of $A(x)$ we evaluate it at the end points and critical number:

$$A(0) = 0, \qquad A(600) = 2400 \cdot 600 - 2 \cdot 600^2 = 720,000, \qquad A(1200) = 0$$

The Closed Interval Method gives the maximum value as $A(600) = 720,000$ ft^2 and the dimensions are $x = 600$ ft, $y = 2400 - 2 \cdot 600 = 1200$ ft.

EXAMPLE 5: A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

$$\text{Minimize: } A = 2\pi r^2 + 2\pi r h$$

$$\text{Constraint: } V = \pi r^2 h = 1500$$

We now solve the second equation for h and substitute the result into the first equation to express A as a function of one variable:

$$\pi r^2 h = 1500 \quad \Longrightarrow \quad h = \frac{1500}{\pi r^2} \quad \Longrightarrow \quad A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \frac{1500}{\pi r^2} = 2\pi r^2 + \frac{3000}{r}$$

To find the absolute minimum value of $A = 2\pi r^2 + \dfrac{3000}{r}$, we use the First Derivative Test for Absolute Extreme Values. The derivative of $A(r)$ is

$$A'(r) = \left(2\pi r^2 + \frac{3000}{r}\right)' = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2} \qquad \text{(see Appendix for details)}$$

Since $r > 0$, the only critical number is $r = \sqrt[3]{\dfrac{3000}{4\pi}} = \sqrt[3]{\dfrac{750}{\pi}}$. It is easy to see that $A'(r) < 0$ for all $0 < r < \sqrt[3]{\dfrac{750}{\pi}}$ and $A'(r) > 0$ for all $r > \sqrt[3]{\dfrac{750}{\pi}}$. Therefore the minimum value of the area must occur at $r = \sqrt[3]{\dfrac{750}{\pi}} \approx 6.2035$ cm and this value is

$$A\left(\sqrt[3]{\frac{750}{\pi}}\right) \approx 725.3964 \text{ cm}^2$$

Finally, the height of the can is

$$h = \frac{1500}{\pi r^2} = \frac{1500}{\pi (750/\pi)^{2/3}} = 2r \approx 12.4070 \text{ cm}$$