Chapter 4

Trusses

4.1 Introduction

A typical plane truss is shown in Figure 4.1 below:

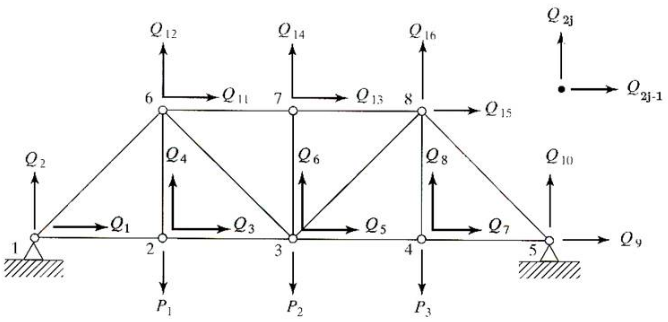

Figure 1:

Every truss element is in direct tension or compression. In a truss, it is required that all loads and reaction are applied only at the joints and that all members are connected together at their ends by frictionless pin joints. In a course on statics, trusses were analyzed using the method of joints and the method of sections. These methods become tedious when applied to large-scale statically indeterminate truss structure.

4.2 Plane Trusses

4.2.1 Local coordinate system and global coordinate system.

The main differences between the one-dimensional structures and trusses is that the element of a truss have various orientations. To accounts for these different orientations, local and global coordinate systems are introduced as follows:

A typical two-dimensional truss element is shown in Figure 4.2 below:

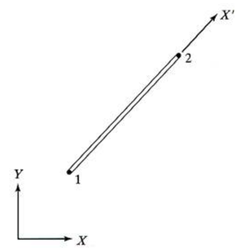

Figure 2:

In the local numbering scheme, the two nodes of the element are numbered 1 and 2. The local coordinate system consists of the x'-axis, which runs along the element from node 1 toward node 2.

The global coordinate system is shown in Figure 4.3 below. The global x, y coordinate system is fixed and does not depend on the orientation of the element. Note that x, y and z form a right-handed coordinate system with the z-axis coming straight out of the paper. In the global coordinate system, every node has two degrees of freedom.

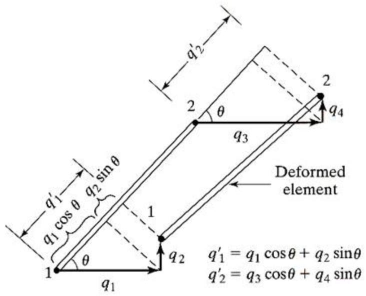

Figure 3:

A systematic numbering scheme is adopted here: A node whose global node number is j has associated with it dofs $2j - 1$ and $2j$. Further, the global displacement associated with node j are Q_{2j-1} and Q_{2j} , as shown in Figure 4.1.

Let $q_1'$and q_2'be the displacements of nodes 1 and 2 in the local coordinate system. Thus, the element displacement vector in the local coordinate system is denoted as:

$$\mathrm{q}' = [q_1', q_2']^{\mathbf{T}}$$

The element displacement vector in the global coordinate system is a (4×1)vector denoted by:

$$\mathrm{q} = [q_1, q_2, q_3, q_4]^{\mathbf{T}}$$

Referring to Figure 4.3, the relationship between $q_1'$and q_2' as follows:

$$q_1' = [q_1 \cos\theta + q_2 \sin\theta]^{\mathbf{T}}$$

$$q_2' = [q_3 \cos\theta + q_4 \sin\theta]^{\mathbf{T}}$$

In matrix form as

$$\mathrm{q}' = \mathrm{Lq}$$

where the transformation matrix L is given by:

$$\mathrm{L} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}$$

Formulas for calculating $\ell$and m is shown in Figure 4.4 below:

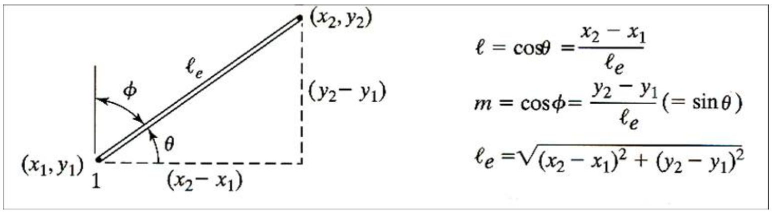

Figure 4:

4.2.2 Element Stiffness Matrix

Element stiffness matrix, k′ for a truss element in the local coordinate system is given by:

$$k' = \frac{E_e A_e}{\ell_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where A_e is the element cross-sectional area and E_e is Young's modulus.

Element stiffness matrix, k in global coordinate system is given by:

$$k = \frac{A_e E_e}{\ell_e}\begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix}$$

4.2.3 Stress Calculations

The stress , in a truss element is given by:

$$\sigma = E_e \epsilon$$

Since the strain $\varepsilon$is the change in length per unit original length,

$$\sigma = E_e \frac{q'_2 - q'_1}{\ell_e}$$

$$\sigma = \frac{Ee}{\ell e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} q'_2 \\ q'_1 \end{Bmatrix}$$

This equation can be written in terms of the global displacement, **q** using the transformation q′ = Lqas

$$\sigma = \frac{Ee}{\ell e} \begin{bmatrix} -1 & 1 \end{bmatrix} \mathrm{Lq}$$

Substituting for **L** yields

$$\mathrm{L} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}$$

Thus,

$$\sigma = \frac{Ee}{\ell e} \begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix} \mathrm{q}$$

Example:

Consider the four-bar truss shown in Figure 4.5 below. It is given that $E = 29.5 \times 10^6 psi$ and $A_e = 1\,in^2$ for all elements.

a) Determine the element stiffness matrix for each element.

b) Assemble the structural stiffness matrix **K** for entire truss.

c) Using the elimination approach, solve for the nodal displacement.

d) Recover the stresses in each element.

e) Calculate the reaction forces.

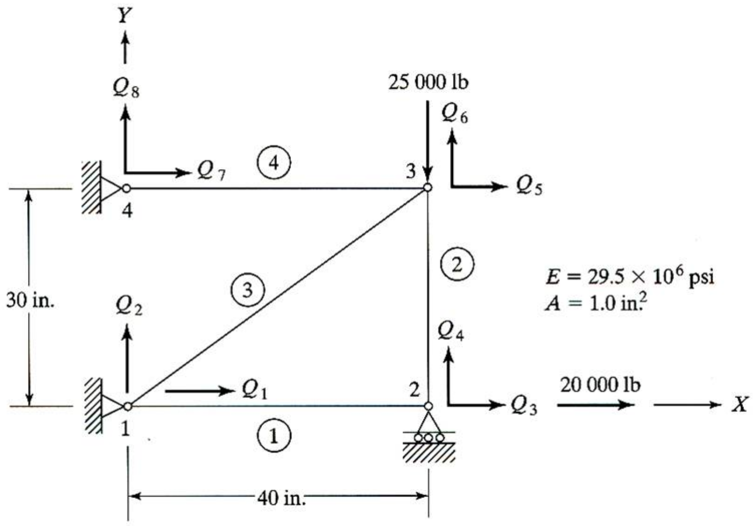

Figure 5:

Solution:

a) Nodal coordinate data and element information is:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivity of element 2 can be defined as 2-3 instead of 3-2 as in the previous table.

Using formulas in Figure 4.4, together with the nodal coordinate data and the given element connectivity information, we obtain the direction cosines table:

Element	ℓ_e	ℓ	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of element 3 are obtained as $\ell = \frac{(x_3 - x_1)}{\ell_e} = \frac{(40-0)}{50} = 0.8$ and $m = \frac{(y_3 - y_1)}{\ell_e} = \frac{(30-0)}{50} = 0.6$

The element stiffness matrices for element 1, 2, 3 and 4 are given by:

$$\mathrm{k}^1 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{k}^2 = \frac{29.5 \times 10^6}{30} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{k}^3 = \frac{29.5 \times 10^6}{50} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$\mathrm{k}^4 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The structural stiffness matrix **K** is now assembled from the element stiffness matrices.

$$K = \frac{29.5\times10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.23 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c) The elimination approach will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7 and 8 which corresponding to fixed support are deleted from the K matrix. The reduced finite element equations are given as:

$$\frac{29.5\times10^6}{40}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix}\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20000 \\ 0 \\ -25000 \end{Bmatrix}$$

Solution of these equations yields the displacements:

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12\times10^{-3} \\ 5.65\times10^{-3} \\ -22.25\times10^{-3} \end{Bmatrix} \text{in}$$

The nodal displacement vector for the entire structure can therefore be written as:

$$Q = \begin{bmatrix} 0 & 0 & 27.12\times10^{-3} & 0 & 5.65\times10^{-3} & -22.25\times10^{-3} & 0 & 0 \end{bmatrix}^T \text{in}$$

d) The stress in each element can now be determined from the equation:

$$\sigma = \frac{Ee}{\ell e}\begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix}\text{q}$$

The connectivity of element 1 is 1-2. The nodal displacement vector for element 1 is given by:

$$q = \begin{bmatrix} 0 & 0 & 27.12\times10^{-3} & 0 \end{bmatrix}^T$$

Stress equation for element 1 becomes:

$$\sigma_1 = \frac{29.5\times10^6}{40}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \end{Bmatrix} = 20000\ Psi$$

Stress in element 2 is given by:

$$\sigma_2 = \frac{29.5\times10^6}{30}\begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}\begin{Bmatrix} 5.65\times10^{-3} \\ -22.25\times10^{-3}0 \\ -27.12\times10^{-3} \\ 0 \end{Bmatrix} = -21880\ Psi$$

Following similar steps, we get:

$\sigma_3 = 5208\ Psi$

$\sigma_4 = 4167\ Psi$

e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7 and 8 which corrsespond to fixed support. These are obtained by substituting for Q into the original finite element equation R=KQ-F. In this substitution, only those rows of K corresponding to the support dofs are needed and F=0 for these dofs. Thus, we have,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5\times10^6}{30}\begin{bmatrix} 22.68 & 5.76 & -15 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \\ 5.65\times10^{-3} \\ -22.25\times10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

which results in:

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} lb$$

A free body diagram of the truss with reaction forces and applied loads is shown in Figure 4.6 below.

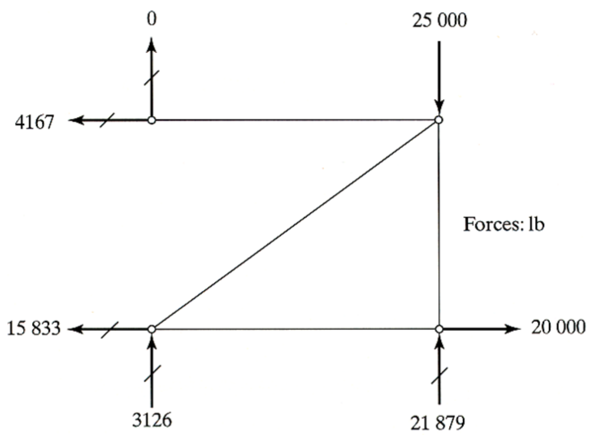

Figure 6:

Problem above can be solve using Truss2D program:

Input data for Truss2D program:

<< 2D TRUSS ANALYSIS USING BAND SOLVER>>
EXAMPLE 4.1

NN	NE	NM	NDIM	NEN	NDN	
4	4	1	2	2	2	
ND	NL	NMPC				
5	2	0				
Node#	X	Y				
1	0	0				
2	40	0				
3	40	30				
4	0	30				
Elem#	N1	N2	Mat#	Area	TempRise	
1	1	2	1	1	0	0
2	3	2	1	1	0	0
3	3	1	1	1	0	0
4	4	3	1	1	0	0
DOF#	Displacement					
1	0					
2	0					
4	0					
7	0					
8	0					
DOF#	Load					
3	20000					
6	-25000					
MAT#	E	Alpha				
1	2.95E+07	1.20E-05				
	B1		i	B2	j	B3 (Multi-point constr. B1*Qi+B2*Qj=B3)

Results from Truss2D program:

Results from Program TRUSS2D
EXAMPLE 4.1

Node#	X-Displ	Y-Displ
1	1.3241E-06	-2.6E-07
2	0.02711997	-1.8E-06
3	0.0056507	-0.02225
4	3.485E-07	0
Elem#	Stress	
1	20000	
2	-21874.647	
3	-5208.921	
4	4167.13684	
DOF#	Reaction	
1	-15832.863	
2	3125.35263	
4	21874.6474	
7	-4167.1368	
8	0	

4.2.4 Temperature Effects

The element temperature load in the local coordinate system is given by

$$\theta' = E_e A_e \epsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Where initial strain associated with a temperature change is given by $\epsilon_0 = \alpha \Delta T$, in which is the coefficient of thermal expansion and ΔT is the average change in temperature in the element.

The initial strain ϵ_0 can also be induced by forcing members into places that are either too long or too short, due to fabrication errors.

The load vector in the global coordinate system, is given by:

$$\theta = L^{\mathrm{T}} \theta'$$

Substituting for L, where

$$L = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}$$

we get

$$\theta^e = E_e A_e \epsilon_0 \begin{Bmatrix} -\ell \\ -m \\ \ell \\ m \end{Bmatrix}$$

The temperature load, along with other externally applied loads, are assembled in the usual manner to obtain the nodal load vector F. Once the displacement are obtained by solving the finite element equations, the stress in each truss element is obtained from:

$$\sigma = E\left(\epsilon - \epsilon_0\right)$$

This equation can also be written as:

$\sigma_e = \frac{E_e}{\ell_e}\begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix} \mathrm{q} - E_e \alpha \Delta T$

Example:

The four-bar truss of Figure 4.5 is considered here but the loading is different. Take $E=29.5\times10^6$psi and $\alpha=1/150000$ per °F.

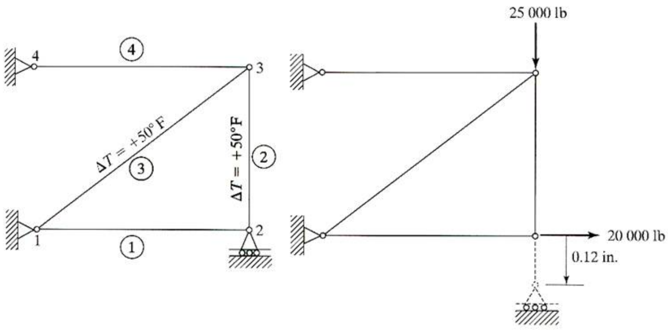

Figure 7:

a) There is an increase in temperature of 50 °F in bars 2 and 3. No others loads on the structure. Determine the nodal displacement and element stresses as a result of this temperature increase. Use the elimination approach.

b) A support settlement effect is considered here. Node 2 settles by 0.12 in vertically down and in addition, two point loads are applied on the structure. Write down(without solving) the equilibrium equations $KQ = F$ where K and F are the modified structural stiffness matrix and load vector. Use the penalty approach.

c) Use the program Truss2D to obtain the solution to part(b).

Solution:

a) The stiffness matrix for the truss structure has already been developed in Example 4.1. Only the load vector needs to be assembled due to the temperature increase.

Using equation,

$$\theta^e = E_e A_e \epsilon_0 \begin{Bmatrix} -\ell \\ -m \\ \ell \\ m \end{Bmatrix}$$

Temperature load due to temperature increase for element 2 and 3 :

$$\theta^2 = \frac{29.5 \times 10^6 \times 50}{150000} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}$$

$$\theta^3 = \frac{29.5 \times 10^6 \times 50}{150000} \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix}$$

The θ^2 and θ^3 vectors contribute to the global load vector F. Using the elimination approach, we can delete all rows and columns corresponding to support dofs in K and F. The resulting finite element equations are:

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15.0 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 7866.7 \\ 15733.3 \end{Bmatrix}$$

which yield,

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.003951 \\ 0.01222 \end{Bmatrix} \text{in}$$

The element stresses can now be obtained from equation

$$\sigma_e = \frac{E_e}{\ell_e} \begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix} \text{q} - E_e \alpha \Delta T$$

For example, the stress in element 2 is given as:

$$\sigma_2 = \frac{29.5 \times 10^6}{30} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 0.003951 \\ 0.01222 \\ 0 \\ 0 \end{Bmatrix} - \frac{29.5 \times 10^6 \times 50}{150000} = 2183 \, Psi$$

The complete stress solution is:

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{array}\right\} = \left\{\begin{array}{c}0\\ 2183\\ -3643\\ 2914\end{array}\right\} Psi$$

Solution using Truss2D program:

Input data for Truss 2D program

<< 2D TRUSS ANALYSIS USING BAND SOLVER>>						
EXAMPLE 4.1						
NN	NE	NM		NDIM	NEN	NDN
4	4	1		2	2	2
ND	NL	NMPC				
5	2	0				
Node#	X	Y				
1	0	0				
2	40	0				
3	40	30				
4	0	30				
Elem#	N1	N2		Mat#	Area	TempRise
1	1	2		1	1	0
2	3	2		1	1	50
3	3	1		1	1	50
4	4	3		1	1	0
DOF#	Displacement					
1	0					
2	0					
4	0					
7	0					
8	0					
DOF#	Load					
3	0					
6	0					
MAT#	E	Alpha				
1	2.95E+07	6.67E-06				
	B1	i	B2	j	B3	(Multi-point constr. B1*Qi+B2*Qj=B3)

Results from Truss2D program:

```
Results from Program TRUSS2D
EXAMPLE 4.1
Node#        X-Displ          Y-Displ
        1       -2.4E-07       -1.8E-07
        2       -2.4E-07       1.83E-07
        3       0.003951       0.012222
        4       2.44E-07              0
Elem#        Stress
        1       -7.8E-17
        2       2185.061
        3       -3641.77
        4       2913.414
DOF#         Reaction
        1       2913.414
        2       2185.061
        4       -2185.06
        7       -2913.41
        8              0
```

b) Support 2 settles by 0.12 in vertically down and two concentrated forces are applied as shown in Figure 4.7(b). In the penalty approach, a large spring constant c is added to the diagonal elements in the structural stiffness matrix at those dofs where the displacement are specified. Typically, c may be chosen 104 times the largest diagonal element of the unmodified stiffness matrix. Further, a force Ca is added to the force vector, where a is the specified displacement. In this example, for dof 4, a=-0.12 and consequently a force equal to –0.12C gets added to the fourth location in the force vector. The modified finite element equations are given by:

$$\frac{29.5\times10^6}{600}\begin{bmatrix} 22.68+C & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ & 4.32+C & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ & & 15.0 & 0 & 0 & 0 & 0 & 0 \\ & & & 20.0+C & 0 & -20.0 & 0 & 0 \\ & & & & 22.68 & 5.76 & -15.0 & 0 \\ & & & & & 24.32 & 0 & 0 \\ & & & & & & 15.0+C & 0 \\ simetri & & & & & & & 0 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20000 \\ -0.12C \\ 0 \\ -25000 \\ 0 \\ 0 \end{Bmatrix}$$

c) Solution using TRUSS2D program:

Input data for Truss2D program:

<< 2D TRUSS ANALYSIS USING BAND SOLVER>>
EXAMPLE 4.1

NN	NE	NM	NDIM	NEN	NDN
4	4	1	2	2	2
ND	NL	NMPC			
5	2	0			
Node#	X	Y			
1	0	0			
2	40	-0.12			
3	40	30			
4	0	30			
Elem#	N1	N2	Mat#	Area	TempRise
1	1	2	1	1	0
2	3	2	1	1	50
3	3	1	1	1	50
4	4	3	1	1	0
DOF#	Displacement				
1	0				
2	0				
4	-0.12				
7	0				
8	0				
DOF#	Load				
3	20000				
6	-25000				
MAT#	E	Alpha			
1	2.95E+07	6.67E-06			
	B1	i	B2	j	B3(Multi-point constr. B1*Qi+B2*Qj=B3)

Result from Truss2D program:

Results from Program TRUSS2D
EXAMPLE 4.1

Node#	X-Displ	Y-Displ
1	-5.7E-07	-1.7E-06
2	0.026758	-0.12
3	0.036266	-0.115
4	2.24E-06	0
Elem#	Stress	
1	20000.09	
2	-4941.79	
3	-33430.4	
4	26744.28	
DOF#	Reaction	
1	6744.283	
2	20118.21	
4	4881.788	
7	-26744.3	
8	0	